

MULTILOOP LIE ALGEBRAS AND THE CONSTRUCTION OF EXTENDED AFFINE LIE ALGEBRAS

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ABSTRACT. It is known that a multiloop Lie algebra, which is constructed using multiloop realization, can be a Lie \mathbb{Z}^n -torus if the given multiloop Lie algebra satisfies several conditions, and it is also known that a family of extended affine Lie algebras (EALAs) are obtained from a Lie \mathbb{Z}^n -torus. In many cases, however, even if a given multiloop Lie algebra does not satisfy these conditions, we can also construct a family of EALAs from it. In this paper, we study this construction, and prove that two families of EALAs constructed from two multiloop Lie algebras are coincide up to isomorphisms as EALAs if and only if two multiloop Lie algebras are “support-isomorphic”. Also, we give a necessary and sufficient condition for two multiloop Lie algebras to be support-isomorphic.

1. INTRODUCTION

The multiloop realization is introduced in [ABFP1]: from an algebra \mathcal{A} that is not necessarily associative or unital, a finite sequence of mutually commutative finite order automorphisms $\sigma = (\sigma_1, \dots, \sigma_n)$, and a sequence of positive integers $\mathbf{m} = (m_1, \dots, m_n)$ such that $\sigma_i^{m_i} = \text{id}$ for $1 \leq i \leq n$, we can construct a \mathbb{Z}^n -graded algebra $M_{\mathbf{m}}(\mathcal{A}, \sigma)$ called a multiloop algebra.

We consider the case where in the above realization an algebra \mathcal{A} is a finite dimensional simple Lie algebra \mathfrak{g} . In this case since $\mathfrak{g}^\sigma := \{g \in \mathfrak{g} \mid \sigma_i(g) = g \text{ for all } i\} \neq \{0\}$ is reductive, we can consider a root space decomposition of $M_{\mathbf{m}}(\mathfrak{g}, \sigma)$ with respect to a Cartan subalgebra \mathfrak{h} in \mathfrak{g}^σ , and then we can see $M_{\mathbf{m}}(\mathfrak{g}, \sigma)$ as a $Q_{\mathfrak{h}} \times \mathbb{Z}^n$ -graded Lie algebra where $Q_{\mathfrak{h}}$ is a root lattice. In this paper, we call the $Q_{\mathfrak{h}} \times \mathbb{Z}^n$ -graded Lie algebra a multiloop Lie algebra, and denote it by $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$. In [ABFP2], the authors have proved that $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ can be a Lie \mathbb{Z}^n -torus if σ satisfies some conditions (the principal condition is that \mathfrak{g}^σ is a simple Lie algebra), and in that case it is called a multiloop Lie \mathbb{Z}^n -torus. A Lie \mathbb{Z}^n -torus is a $Q \times \mathbb{Z}^n$ -graded Lie algebra, where Q is a root lattice of an irreducible finite root system, satisfying several axioms. E. Neher has proved in [N] that if a centreless Lie \mathbb{Z}^n -torus is given, we can construct a family of extended affine Lie algebras (EALAs, for short). However, unless $\mathfrak{g}^\sigma = \{0\}$, we can construct a family of EALAs from $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ even if σ does not satisfy the condition for $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ to be a Lie \mathbb{Z}^n -torus. This fact can be seen by proving that the $Q_{\mathfrak{h}}$ -support of $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ with respect to \mathfrak{h} is an irreducible finite root system. In this paper, we study this construction of a family of EALAs from a multiloop Lie algebra.

In [AF], it has been proved that there exists a one-to-one correspondence between centreless Lie \mathbb{Z}^n -tori up to isotopy and families of EALAs up to isomorphism, where isotopy is an equivalence relation on a class of Lie \mathbb{Z}^n -tori defined in [ABFP2]. In

this article, we see that the similar result is obtained in the case of multiloop Lie algebras; we define an equivalence relation “support-isomorphic” on the class of multiloop Lie algebras (see Definition 2.2.1), and then we prove that two families of EALAs constructed from two multiloop Lie algebras coincide up to isomorphism if and only if two multiloop Lie algebras are support-isomorphic. Also, we give a necessary and sufficient condition for two multiloop Lie algebras to be support-isomorphic.

As we prove in Theorem 5.1.4, a multiloop Lie algebra $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ is support-isomorphic to some Lie \mathbb{Z}^n -torus if and only if $\mathfrak{g}^\sigma \neq \{0\}$. From this fact, we can see that the class of EALAs which can be constructed from multiloop Lie algebras coincides with that constructed from multiloop Lie \mathbb{Z}^n -tori. It is, however, expected that, at least in some cases, considering whole multiloop Lie algebras makes it easy to study the classification problem of EALAs.

We briefly outline the contents of this article. In section 2, we recall the definition and some results about multiloop algebras, and define a support-isomorphism. In section 3, we define a multiloop Lie algebra $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$, and observe the property of the support of a $Q_{\mathfrak{h}}$ -grading. In section 4, we study a support-isomorphism of multiloop Lie algebras. In section 5, we give a necessary and sufficient condition for multiloop Lie algebra to be support-isomorphic to some Lie \mathbb{Z}^n -torus, and finally, we study the construction of EALAs from a multiloop Lie algebra.

Assumptions and Notation.

- (a) Throughout this paper all vector spaces and algebras are defined over a base field k of characteristic 0 and we assume that k is *algebraically closed*. In this paper an algebra is not necessarily associative or unital.
- (b) For each $n \in \mathbb{Z}_{>0}$, we choose a primitive n -th root of unity $\zeta_n \in k$ satisfying the following condition: for all $m, n \in \mathbb{Z}_{>0}$,

$$\zeta_{mn}^m = \zeta_n. \quad (1)$$

- (c) For an n -tuple of positive integers $\mathbf{m} = (m_1, \dots, m_n)$, let

$$\bar{\Lambda}_{\mathbf{m}} = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_n\mathbb{Z}.$$

- (d) For a group Λ and a subset $S \subseteq \Lambda$, let $\langle S \rangle$ be a subgroup of Λ generated by S .
- (e) If $\mathcal{B} = \bigoplus_{\lambda \in \Lambda} \mathcal{B}^\lambda$ is a Λ -graded algebra for some abelian group Λ , we put

$$\text{supp}_\Lambda(\mathcal{B}) = \{\lambda \in \Lambda \mid \mathcal{B}^\lambda \neq \{0\}\} \subseteq \Lambda.$$

2. MULTILOOP ALGEBRAS

Although we are interested only in Lie algebras, we deal with general algebras in this section.

2.1. Definitions and some results. First, we recall the following basic definitions.

Definition 2.1.1. Suppose that \mathcal{A} is an algebra.

- (a) Let $C(\mathcal{A})$ be the subalgebra of $\text{End}_k(\mathcal{A})$ consisting of the k -linear endomorphisms of \mathcal{A} that commute with all left and right multiplications by elements of \mathcal{A} . We call $C(\mathcal{A})$ the *centroid* of \mathcal{A} .
- (b) We say \mathcal{A} is *central-simple* if \mathcal{A} is simple and $C(\mathcal{A}) = k \cdot \text{id}$.

Definition 2.1.2. Let Λ be an abelian group and $\mathcal{B} = \bigoplus_{\lambda \in \Lambda} \mathcal{B}^\lambda$ be a Λ -graded algebra.

- (a) We say \mathcal{B} is *graded-simple* if $\mathcal{B}\mathcal{B} \neq \{0\}$ and graded ideals of \mathcal{B} are only $\{0\}$ and \mathcal{B} .
- (b) Suppose that \mathcal{B} is graded-simple. Then $C(\mathcal{B}) = \bigoplus_{\lambda \in \Lambda} C(\mathcal{B})^\lambda$ is a unital commutative associative Λ -graded algebra where

$$C(\mathcal{B})^\lambda = \{c \in C(\mathcal{B}) \mid c\mathcal{B}^\mu \subseteq \mathcal{B}^{\lambda+\mu} \text{ for } \mu \in \Lambda\},$$

and $\Gamma_\Lambda(\mathcal{B}) := \text{supp}_\Lambda(C(\mathcal{B}))$ is a subgroup of Λ [BN, Proposition 2.16]. We call $\Gamma_\Lambda(\mathcal{B})$ the *central grading group* of \mathcal{B} . We say \mathcal{B} is *graded-central-simple* if \mathcal{B} is graded-simple and $C(\mathcal{B})^0 = k \cdot \text{id}$.

Definition 2.1.3. Let Λ, Λ' be abelian groups.

- (a) Suppose that \mathcal{B} and \mathcal{B}' are Λ -graded algebras. Then we say \mathcal{B} and \mathcal{B}' are *Λ -graded-isomorphic* if there exists an algebra isomorphism $\varphi : \mathcal{B} \rightarrow \mathcal{B}'$ such that

$$\varphi(\mathcal{B}^\lambda) = \mathcal{B}'^\lambda$$

for $\lambda \in \Lambda$. In that case, we call φ a *Λ -graded-isomorphism*, and we write $\mathcal{B} \cong_\Lambda \mathcal{B}'$.

- (b) Suppose that \mathcal{B} is a Λ -graded algebra and \mathcal{B}' is a Λ' -graded algebra. Then we say \mathcal{B} and \mathcal{B}' are *isograded-isomorphic* if there exist an algebra isomorphism $\varphi : \mathcal{B} \rightarrow \mathcal{B}'$ and a group isomorphism $\varphi_\Lambda : \Lambda \rightarrow \Lambda'$ such that

$$\varphi(\mathcal{B}^\lambda) = \mathcal{B}'^{\varphi_\Lambda(\lambda)}$$

for $\lambda \in \Lambda$. In that case we call φ an *isograded-isomorphism*, and we write $\mathcal{B} \cong_{\text{ig}} \mathcal{B}'$.

To define a multiloop Lie algebra, we use the following notation. Suppose that \mathcal{A} is an algebra. We denote the set of n -tuples of commuting finite order automorphisms of \mathcal{A}

$$\{(\sigma_1, \dots, \sigma_n) \in \text{Aut}(\mathcal{A})^n \mid \sigma_i \sigma_j = \sigma_j \sigma_i, \text{ord}(\sigma_i) < \infty \text{ for all } i, j\}$$

by $\text{Aut}_{\text{cfo}}^n(\mathcal{A})$. For $\sigma = (\sigma_1, \dots, \sigma_n) \in \text{Aut}_{\text{cfo}}^n(\mathcal{A})$, we put

$$\mathcal{A}^\sigma = \{u \in \mathcal{A} \mid \sigma_i(u) = u \text{ for } 1 \leq i \leq n\},$$

and we write $\text{ord}(\sigma) = (\text{ord}(\sigma_1), \dots, \text{ord}(\sigma_n)) \in \mathbb{Z}_{>0}^n$.

A multiloop Lie algebra has been defined in [ABFP1] as follows:

Definition 2.1.4. Suppose that \mathcal{A} is an algebra. Let $n \in \mathbb{Z}_{>0}$, and assume that $\sigma = (\sigma_1, \dots, \sigma_n) \in \text{Aut}_{\text{cfo}}^n(\mathcal{A})$ and $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{>0}^n$ satisfy

$$\sigma_i^{m_i} = \text{id} \text{ for } 1 \leq i \leq n.$$

(Henceforth, we write $\sigma^{\mathbf{m}} = \mathbf{id}$ to denote this condition). Note that we do not necessarily assume that each m_i is an order of σ_i . For $\lambda = (l_1, \dots, l_n) \in \mathbb{Z}^n$, let

$$\bar{\lambda} = (\bar{l}_1, \dots, \bar{l}_n) \in \bar{\Lambda}_{\mathbf{m}} (= \mathbb{Z}/m_1\mathbb{Z} \times \dots \times \mathbb{Z}/m_n\mathbb{Z})$$

be the image of λ under the canonical group homomorphism from \mathbb{Z}^n onto $\bar{\Lambda}_{\mathbf{m}}$. Using σ and \mathbf{m} , we define a $\bar{\Lambda}_{\mathbf{m}}$ -grading on \mathcal{A} as follows: for $\bar{\lambda} = (\bar{l}_1, \dots, \bar{l}_n) \in \bar{\Lambda}_{\mathbf{m}}$,

$$\mathcal{A}^{\bar{\lambda}(\sigma, \mathbf{m})} := \{u \in \mathcal{A} \mid \sigma_i(u) = \zeta_{m_i}^{l_i} u \text{ for } 1 \leq i \leq n\}. \quad (2)$$

(We usually use a notation $\mathcal{A}^{\bar{\lambda}}$ instead of $\mathcal{A}^{\bar{\lambda}(\sigma, \mathbf{m})}$ when it is obvious from the context that \mathcal{A} is graded by σ and \mathbf{m}). Then we can define a \mathbb{Z}^n -graded algebra

$$M_{\mathbf{m}}(\mathcal{A}, \sigma) = \bigoplus_{\lambda \in \mathbb{Z}^n} \mathcal{A}^{\bar{\lambda}} \otimes t^\lambda \subseteq \mathcal{A} \otimes k[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \quad (3)$$

where for $\lambda = (l_1, \dots, l_n)$, we put $t^\lambda = t_1^{l_1} t_2^{l_2} \dots t_n^{l_n}$. We call the \mathbb{Z}^n -graded algebra $M_{\mathbf{m}}(\mathcal{A}, \sigma)$ the *multiloop algebra* of σ (based on \mathcal{A} and relative to \mathbf{m}). We call n the *nullity* of $M_{\mathbf{m}}(\mathcal{A}, \sigma)$.

By [ABFP1, Proposition 8.2.2], we have the following:

Lemma 2.1.5. *Suppose that \mathcal{A} is a central-simple algebra, and $\mathcal{B} = M_{\mathbf{m}}(\mathcal{A}, \sigma)$ is a multiloop algebra of $\sigma \in \text{Aut}_{\text{cfo}}^n(\mathcal{A})$ relative to $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{>0}^n$ where $\sigma^{\mathbf{m}} = \text{id}$. Then \mathcal{B} is a graded-central-simple \mathbb{Z}^n -graded algebra, and*

$$\Gamma_{\mathbb{Z}^n}(\mathcal{B}) = m_1 \mathbb{Z} \times \dots \times m_n \mathbb{Z} \subseteq \mathbb{Z}^n.$$

where $\Gamma_{\mathbb{Z}^n}(\mathcal{B})$ is the central grading group of \mathcal{B} . In particular, the rank of $\Gamma_{\mathbb{Z}^n}(\mathcal{B})$ is n . \square

We use the following notation. Let \mathcal{A} be an algebra and $\sigma = (\sigma_1, \dots, \sigma_n) \in \text{Aut}_{\text{cfo}}^n(\mathcal{A})$. For $P = (p_{ij}) \in \text{GL}_n(\mathbb{Z})$, we set

$$\sigma^P = (\prod_{1 \leq i \leq n} \sigma_i^{p_{i1}}, \prod_{1 \leq i \leq n} \sigma_i^{p_{i2}}, \dots, \prod_{1 \leq i \leq n} \sigma_i^{p_{in}}).$$

Since σ_i 's commute with each other and each σ_i has a finite order, $\sigma^P \in \text{Aut}_{\text{cfo}}^n(\mathcal{A})$. It is easy to check that $(\sigma^P)^Q = \sigma^{PQ}$ for $P, Q \in \text{GL}_n(\mathbb{Z})$. Therefore, $P : \sigma \mapsto \sigma^P$ defines a right $\text{GL}_n(\mathbb{Z})$ -action on $\text{Aut}_{\text{cfo}}^n(\mathcal{A})$. If \mathcal{A}' is another algebra and $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$ is an algebra isomorphism, we write

$$\varphi \sigma \varphi^{-1} = (\varphi \sigma_1 \varphi^{-1}, \dots, \varphi \sigma_n \varphi^{-1}) \in \text{Aut}_{\text{cfo}}^n(\mathcal{A}').$$

The following definition is introduced in [ABFP1, Definition 8.1.1] (in the definition, we let $\text{diag}(a_1, \dots, a_n)$ denote an n -diagonal matrix with the diagonal entries (a_1, \dots, a_n)):

Definition 2.1.6. For $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{>0}^n$ and $\mathbf{m}' = (m'_1, \dots, m'_n) \in \mathbb{Z}_{>0}^n$, we set $D_{\mathbf{m}} = \text{diag}(m_1, \dots, m_n)$, $D_{\mathbf{m}'} = \text{diag}(m'_1, \dots, m'_n)$. For $P \in \text{GL}_n(\mathbb{Z})$, we say that P is $(\mathbf{m}', \mathbf{m})$ -admissible if $D_{\mathbf{m}'} {}^t P D_{\mathbf{m}}^{-1} \in \text{GL}_n(\mathbb{Z})$ where ${}^t P$ is a transpose of P .

Proposition 2.1.7. *Suppose that \mathcal{A} and \mathcal{A}' are central-simple algebras. Assume that $\sigma \in \text{Aut}_{\text{cfo}}^n(\mathcal{A})$, $\sigma' \in \text{Aut}_{\text{cfo}}^n(\mathcal{A}')$ and $\mathbf{m}, \mathbf{m}' \in \mathbb{Z}_{>0}^n$ satisfy $\sigma^{\mathbf{m}} = \text{id}$, $\sigma'^{\mathbf{m}'} = \text{id}$. Let $\mathcal{B} = M_{\mathbf{m}}(\mathcal{A}, \sigma)$, $\mathcal{B}' = M_{\mathbf{m}'}(\mathcal{A}', \sigma')$. Then the following two statements are equivalent:*

- (a) $\mathcal{B} \cong_{\text{ig}} \mathcal{B}'$.
- (b) *There exist a matrix $P \in \text{GL}_n(\mathbb{Z})$ and an algebra isomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$ such that P is $(\mathbf{m}', \mathbf{m})$ -admissible and*

$$\sigma' = \varphi \sigma^P \varphi^{-1}. \quad (4)$$

Moreover, if P and φ satisfy (b), we can take an isograded-isomorphism $\psi : \mathcal{B} \rightarrow \mathcal{B}'$ satisfying $\psi(x \otimes 1) = \varphi(x) \otimes 1$ for $x \in \mathcal{A}^\sigma$.

Proof. The first statement is [ABFP1, Theorem 8.3.2 (ii)]. Suppose that P, φ satisfy (b), and let $Q = D_{\mathbf{m}'}^t P D_{\mathbf{m}}^{-1} \in \mathrm{GL}_n(\mathbb{Z})$. If we define $\psi : \mathcal{B} \rightarrow \mathcal{B}'$ as

$$\mathcal{B}^\lambda \ni x \otimes t^\lambda \mapsto \varphi(x) \otimes t^{\lambda^t Q} \in \mathcal{B}'^{\lambda^t Q}$$

for $\lambda \in \mathbb{Z}^n, x \in \mathcal{A}^{\bar{\lambda}(\sigma, \mathbf{m})}$, then ψ is an isograded-isomorphism by [ABFP1, Proposition 8.2.1]. It is clear that $\psi(x \otimes 1) = \varphi(x) \otimes 1$ for $x \in \mathcal{A}^\sigma$. \square

2.2. Support-isomorphism. Let \mathcal{B} be a Λ -graded algebra for an abelian group Λ , and take a subgroup $\Lambda_{\mathrm{sub}} \subseteq \Lambda$ such that $\langle \mathrm{supp}_\Lambda(\mathcal{B}) \rangle \subseteq \Lambda_{\mathrm{sub}}$. Since $\mathcal{B} = \bigoplus_{\lambda \in \Lambda_{\mathrm{sub}}} \mathcal{B}^\lambda$, we can consider \mathcal{B} canonically as a Λ_{sub} -graded algebra. In particular, we can see \mathcal{B} as $\langle \mathrm{supp}_\Lambda(\mathcal{B}) \rangle$ -graded.

Definition 2.2.1. Let Λ, Λ' be abelian groups, and suppose that \mathcal{B} is a Λ -graded algebra and \mathcal{B}' is a Λ' -graded algebra. We say \mathcal{B} and \mathcal{B}' are *support-isograded-isomorphic* (or *support-isomorphic*, for short) if there exist an algebra isomorphism $\varphi : \mathcal{B} \rightarrow \mathcal{B}'$ and a group isomorphism $\varphi_{\mathrm{su}} : \langle \mathrm{supp}_\Lambda(\mathcal{B}) \rangle \rightarrow \langle \mathrm{supp}_{\Lambda'}(\mathcal{B}') \rangle$ such that

$$\varphi(\mathcal{B}^\lambda) = \mathcal{B}'^{\varphi_{\mathrm{su}}(\lambda)}$$

for $\lambda \in \langle \mathrm{supp}_\Lambda(\mathcal{B}) \rangle$: in other words, if \mathcal{B} is considered as $\langle \mathrm{supp}_\Lambda(\mathcal{B}) \rangle$ -graded and \mathcal{B}' as $\langle \mathrm{supp}_{\Lambda'}(\mathcal{B}') \rangle$ -graded, then \mathcal{B} and \mathcal{B}' are isograded-isomorphic. In that case, we call φ a *support-isograded-isomorphism* (or *support-isomorphism*, for short), and we write $\mathcal{B} \cong_{\mathrm{supp}} \mathcal{B}'$.

The following Lemma is obvious from the definitions:

Lemma 2.2.2. *Let Λ, Λ' be abelian groups, and suppose that \mathcal{B} is a Λ -graded algebra and \mathcal{B}' is a Λ' -graded algebra.*

- (a) *If $\mathcal{B} \cong_{\mathrm{ig}} \mathcal{B}'$, then $\mathcal{B} \cong_{\mathrm{supp}} \mathcal{B}'$.*
- (b) *If $\langle \mathrm{supp}_\Lambda(\mathcal{B}) \rangle = \Lambda$ and $\langle \mathrm{supp}_{\Lambda'}(\mathcal{B}') \rangle = \Lambda'$, then $\mathcal{B} \cong_{\mathrm{ig}} \mathcal{B}'$ is equivalent to $\mathcal{B} \cong_{\mathrm{supp}} \mathcal{B}'$.* \square

We would like to give a necessary and sufficient condition for two multiloop algebras based on central-simple algebras to be support-isomorphic. To do this, we need the following lemmas.

Lemma 2.2.3. *Let \mathcal{A} be an algebra, $\sigma \in \mathrm{Aut}_{\mathrm{cfo}}^n(\mathcal{A})$. Then there exists $P \in \mathrm{GL}_n(\mathbb{Z})$ such that*

$$\langle \mathrm{supp}_{\mathbb{Z}^n}(M_{\mathrm{ord}(\sigma^P)}(\mathcal{A}, \sigma^P)) \rangle = \mathbb{Z}^n. \quad (5)$$

Proof. Let $G = \langle \{\sigma_1, \dots, \sigma_n\} \rangle$. By [ABFP2, Proposition 5.1.3], there exists $P \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$|G| = \prod_{i=1}^n \mathrm{ord}(\sigma_i^P), \quad (6)$$

where $|G|$ denotes the cardinal number of G and $\sigma^P = (\sigma_1^P, \dots, \sigma_n^P)$. By [ABFP1, Lemma 3.2.4], (6) is equivalent to (5). \square

Lemma 2.2.4. *Let \mathcal{A} be an algebra and $\mathcal{B} = M_{\mathbf{m}}(\mathcal{A}, \sigma)$ be a multiloop algebra of nullity n . Then it follows that*

$$\mathcal{B} \cong_{\mathrm{supp}} M_{\mathrm{ord}(\sigma)}(\mathcal{A}, \sigma).$$

Proof. Let $a_i \in \mathbb{Z}_{>0}$ be a positive number such that $\text{ord}(\sigma_i) = m_i/a_i$ for $1 \leq i \leq n$. We write $\mathcal{B}_{\text{ord}(\sigma)} = M_{\text{ord}(\sigma)}(\mathcal{A}, \sigma)$. Let $f_a : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ be a monomorphism defined as

$$f_a((l_1, \dots, l_n)) = (a_1 l_1, \dots, a_n l_n)$$

for $(l_1, \dots, l_n) \in \mathbb{Z}^n$. By (1), we have $\zeta_{m_i}^{a_i} = \zeta_{\text{ord}(\sigma_i)}$. Using this, for $\lambda = (l_1, \dots, l_n) \in \mathbb{Z}_{>0}^n$,

$$\begin{aligned} \mathcal{A}^{\bar{\lambda}(\sigma, \text{ord}(\sigma))} &= \{u \in \mathcal{A} \mid \sigma_i(u) = \zeta_{\text{ord}(\sigma_i)}^{l_i} u \text{ for } 1 \leq i \leq n\} \\ &= \{u \in \mathcal{A} \mid \sigma_i(u) = \zeta_{m_i}^{a_i l_i} u \text{ for } 1 \leq i \leq n\} \\ &= \mathcal{A}^{\overline{f_a(\lambda)}(\sigma, m)}. \end{aligned} \quad (7)$$

Next, suppose that $\lambda = (l_1, \dots, l_n) \notin \text{Im } f_a$. Then there exists j such that $a_j \nmid l_j$, and then obviously $\mathcal{A}^{\bar{\lambda}(\sigma, m)} = \{0\}$. Consequently, we can define an algebra isomorphism $\varphi : \mathcal{B}_{\text{ord}(\sigma)} \rightarrow \mathcal{B}$ as

$$\mathcal{B}_{\text{ord}(\sigma)}^\lambda = \mathcal{A}^{\bar{\lambda}(\sigma, \text{ord}(\sigma))} \otimes t^\lambda \ni u \otimes t^\lambda \mapsto u \otimes t^{f_a(\lambda)} \in \mathcal{A}^{\overline{f_a(\lambda)}(\sigma, m)} \otimes t^{f_a(\lambda)} = \mathcal{B}^{f_a(\lambda)}$$

for $\lambda \in \mathbb{Z}^n$. Then $f_a(\langle \text{supp}_{\mathbb{Z}^n}(\mathcal{B}_{\text{ord}(\sigma)}) \rangle) = \langle \text{supp}_{\mathbb{Z}^n}(\mathcal{B}) \rangle$, and the above isomorphism is indeed a support-isomorphism. \square

Proposition 2.2.5. *Suppose that $\mathcal{A}, \mathcal{A}'$ are central-simple algebras, and let $\mathcal{B} = M_m(\mathcal{A}, \sigma)$ and $\mathcal{B}' = M_{m'}(\mathcal{A}', \sigma')$ be multiloop algebras of nullity n . Then $\mathcal{B} \cong_{\text{supp}} \mathcal{B}'$ if and only if there exist $P \in \text{GL}_n(\mathbb{Z})$ and an algebra isomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$ such that*

$$\sigma' = \varphi \sigma^P \varphi^{-1}. \quad (8)$$

(In particular, it does not depend on m or m' whether or not $\mathcal{B} \cong_{\text{supp}} \mathcal{B}'$). Moreover, if $P \in \text{GL}_n(\mathbb{Z})$ and an isomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$ satisfy (8), then we can take a support-isomorphism $\psi : \mathcal{B} \rightarrow \mathcal{B}'$ satisfying $\psi(x \otimes 1) = \varphi(x) \otimes 1$ for $x \in \mathcal{A}^\sigma$.

Proof. First, we show the “if” part. Let $M = \text{l.c.m}\{m, m'\} \in \mathbb{Z}_{>0}$ be the least common multiple of $2n$ positive numbers $m_1, \dots, m_n, m'_1, \dots, m'_n$, and let $\mathbf{M} = (M, M, \dots, M) \in \mathbb{Z}_{>0}^n$. Obviously, $\sigma^{\mathbf{M}} = \sigma'^{\mathbf{M}} = \text{id}$. By Lemma 2.2.4,

$$\mathcal{B} \cong_{\text{supp}} M_{\text{ord}(\sigma)}(\mathcal{A}, \sigma) \cong_{\text{supp}} M_{\mathbf{M}}(\mathcal{A}, \sigma)$$

and

$$\mathcal{B}' \cong_{\text{supp}} M_{\text{ord}(\sigma')}(\mathcal{A}', \sigma') \cong_{\text{supp}} M_{\mathbf{M}}(\mathcal{A}', \sigma').$$

It is clear from Definition 2.1.6 that P is (\mathbf{M}, \mathbf{M}) -admissible, and then it follows from Proposition 2.1.7 that

$$M_{\mathbf{M}}(\mathcal{A}, \sigma) \cong_{\text{ig}} M_{\mathbf{M}}(\mathcal{A}', \sigma'),$$

in particular $M_{\mathbf{M}}(\mathcal{A}, \sigma) \cong_{\text{supp}} M_{\mathbf{M}}(\mathcal{A}', \sigma')$ by Lemma 2.2.2. Thus, we have $\mathcal{B} \cong_{\text{supp}} \mathcal{B}'$, and the “if” part follows. The second statement of the proposition is easily checked from the above proof of “if” part, using Proposition 2.1.7 and the proof of Lemma 2.2.4. Next, we show the “only if” part. By Lemma 2.2.3, there exist $Q, R \in \text{GL}_n(\mathbb{Z})$ such that

$$\left\langle \text{supp}_{\mathbb{Z}^n} (M_{\text{ord}(\sigma^Q)}(\mathcal{A}, \sigma^Q)) \right\rangle = \mathbb{Z}^n,$$

and

$$\left\langle \text{supp}_{\mathbb{Z}^n} (M_{\text{ord}(\sigma'^R)}(\mathcal{A}', \sigma'^R)) \right\rangle = \mathbb{Z}^n.$$

We abbreviate

$$\mathcal{B}_Q = M_{\text{ord}(\sigma^Q)}(\mathcal{A}, \sigma^Q) \text{ and } \mathcal{B}'_R = M_{\text{ord}(\sigma'^R)}(\mathcal{A}', \sigma'^R).$$

From the “if” part and the assumption, we have $\mathcal{B}_Q \cong_{\text{supp}} \mathcal{B} \cong_{\text{supp}} \mathcal{B}' \cong_{\text{supp}} \mathcal{B}'_R$. Then $\mathcal{B}_Q \cong_{\text{ig}} \mathcal{B}'_R$ by Lemma 2.2.2 (b). From Proposition 2.1.7, there exist $S \in \text{GL}_n(\mathbb{Z})$ and an algebra isomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$ such that $\sigma'^R = \varphi \sigma^{QS} \varphi^{-1}$. Then we have $\sigma' = \varphi \sigma^{QSR^{-1}} \varphi^{-1}$. \square

3. MULTILOOP LIE ALGEBRAS

3.1. Preliminary lemmas. Suppose that \mathfrak{g} is a finite dimensional simple Lie algebra. Note that \mathfrak{g} is central-simple since k is algebraically closed. For $n \in \mathbb{Z}_{>0}$, let $\sigma = (\sigma_1, \dots, \sigma_n) \in \text{Aut}_{\text{cfo}}^n(\mathfrak{g})$ and $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{>0}^n$ satisfying $\sigma^{\mathbf{m}} = \text{id}$. As (2), we define a $\bar{\Lambda}_{\mathbf{m}}$ -grading on \mathfrak{g} as

$$\mathfrak{g}^{\bar{\lambda}} (= \mathfrak{g}^{\bar{\lambda}(\sigma, \mathbf{m})}) := \{g \in \mathfrak{g} \mid \sigma_i(g) = \zeta_{m_i}^{l_i} g \text{ for } 1 \leq i \leq n\} \quad (9)$$

for $\bar{\lambda} = (\bar{l}_1, \dots, \bar{l}_n) \in \bar{\Lambda}_{\mathbf{m}}$. We denote the Killing form on \mathfrak{g} by (\mid) . Recall that the Killing form is non-degenerate on \mathfrak{g} , invariant, and symmetric.

Lemma 3.1.1. $\mathfrak{g}^{\sigma} (= \mathfrak{g}^{\bar{0}})$ is a reductive Lie algebra.

Proof. Take non-zero arbitrary elements $x^{\bar{\lambda}} \in \mathfrak{g}^{\bar{\lambda}}$, $x^{\bar{\mu}} \in \mathfrak{g}^{\bar{\mu}}$ for $\bar{\lambda} = (\bar{l}_1, \dots, \bar{l}_n)$, $\bar{\mu} = (\bar{r}_1, \dots, \bar{r}_n) \in \bar{\Lambda}_{\mathbf{m}}$. Since (\mid) is preserved by any automorphisms, we have

$$(x^{\bar{\lambda}} \mid x^{\bar{\mu}}) = (\sigma_i(x^{\bar{\lambda}}) \mid \sigma_i(x^{\bar{\mu}})) = \zeta_{m_i}^{l_i + r_i} (x^{\bar{\lambda}} \mid x^{\bar{\mu}})$$

for $1 \leq i \leq n$. This means that

$$(x^{\bar{\lambda}} \mid x^{\bar{\mu}}) = 0 \text{ unless } \bar{\lambda} + \bar{\mu} = \bar{0} \in \bar{\Lambda}_{\mathbf{m}}. \quad (10)$$

In particular, since (\mid) is non-degenerate on \mathfrak{g} , this form is also non-degenerate on \mathfrak{g}^{σ} . If we consider \mathfrak{g} as a representation of \mathfrak{g}^{σ} by an adjoint action, we can see using the definition of the Killing form that \mathfrak{g}^{σ} satisfies the condition (d) of [B1, Chap. 1, §6, Proposition 5], which is equivalent to the condition that \mathfrak{g}^{σ} is reductive. \square

Remark 3.1.2. Note that it is possible that $\mathfrak{g}^{\sigma} = \{0\}$.

Assume that $\mathfrak{g}^{\sigma} \neq \{0\}$. Since \mathfrak{g}^{σ} is reductive, we can take (and fix) a Cartan subalgebra (i.e. a maximal ad-diagonalizable subalgebra) \mathfrak{h} of \mathfrak{g}^{σ} . Note that \mathfrak{h} is not necessarily a Cartan subalgebra of \mathfrak{g} .

Lemma 3.1.3. (a) (\mid) is non-degenerate on \mathfrak{h} .

(b) \mathfrak{h} is ad-diagonalizable on \mathfrak{g} .

Proof. (a) We have the root space decomposition of \mathfrak{g}^{σ} with respect to \mathfrak{h}

$$\mathfrak{g}^{\sigma} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_{\alpha}^{\sigma},$$

where $\mathfrak{g}_{\alpha}^{\sigma} := \{g \in \mathfrak{g}^{\sigma} \mid [h, g] = \langle \alpha, h \rangle g \text{ for } h \in \mathfrak{h}\}$. Note that $\mathfrak{g}_0^{\sigma} = \mathfrak{h}$. For $h \in \mathfrak{h}$, $\alpha, \beta \in \mathfrak{h}^*$ and $x \in \mathfrak{g}_{\alpha}^{\sigma}, y \in \mathfrak{g}_{\beta}^{\sigma}$,

$$\langle \alpha, h \rangle (x \mid y) = ([h, x] \mid y) = -(x \mid [h, y]) = -\langle \beta, h \rangle (x \mid y)$$

since (\mid) is invariant. This means that

$$(x \mid y) = 0 \text{ unless } \alpha + \beta = 0. \quad (11)$$

Hence (a) follows since $(\cdot | \cdot)$ is non-degenerate on \mathfrak{g}^σ as we have shown in the proof of Lemma 3.1.1.

(b) Assume that for some $h \in \mathfrak{h}$, $\text{ad}_{\mathfrak{g}}(h)$ is not ad-diagonalizable. We denote the Jordan decomposition of $\text{ad}_{\mathfrak{g}}(h)$ by

$$\text{ad}_{\mathfrak{g}}(h) = S + T \quad S, T \in \mathfrak{gl}(\mathfrak{g}),$$

where S is the semisimple part and T is the nilpotent part. Then $T \neq 0$. By [H, Lemma 4.2.B], T is a derivation on \mathfrak{g} . Hence, there exists some non-zero element $h_T \in \mathfrak{g}$ such that $\text{ad}_{\mathfrak{g}}(h_T) = T$ since \mathfrak{g} is simple. Due to the property of the Jordan decomposition, there exists a polynomial $f(t) \in k[t]$ such that

$$T = \text{ad}_{\mathfrak{g}}(h_T) = f(\text{ad}_{\mathfrak{g}}(h)), \quad (12)$$

and we have using this that

$$\text{ad}_{\mathfrak{g}}(h_T)(\mathfrak{g}^{\bar{\lambda}}) \subseteq \mathfrak{g}^{\bar{\lambda}} \quad \text{for } \bar{\lambda} \in \bar{\Lambda}_{\mathbf{m}} \quad (13)$$

since $h \in \mathfrak{g}^\sigma$. Thus, we have $h_T \in \mathfrak{g}^\sigma$. From (12), $T|_{\mathfrak{g}^\sigma} = \text{ad}_{\mathfrak{g}^\sigma}(h_T)$ is diagonalizable. Then since $\text{ad}_{\mathfrak{g}^\sigma}(h_T)$ is nilpotent, we have $\text{ad}_{\mathfrak{g}^\sigma}(h_T) = 0$. Hence, we have $h_T \in \mathfrak{h}$, which means that $[z, h_T] = 0$ for all $z \in \mathfrak{h}$. It follows from this and the nilpotency of $\text{ad}_{\mathfrak{g}}(h_T)$ that

$$(z|h_T) = \text{Tr}(\text{ad}_{\mathfrak{g}}(z) \text{ad}_{\mathfrak{g}}(h_T)) = 0 \quad \text{for all } z \in \mathfrak{h}. \quad (14)$$

By Lemma 3.1.3 (a) and (14), we have $h_T = 0$ and this contradicts the assumption. Hence, (b) follows. \square

3.2. The definition of multiloop Lie algebras. In section 2, we have defined a multiloop algebra based on a general algebra. By the abuse of language, we use a term “multiloop Lie algebra” in a different sense from that.

Suppose that \mathfrak{g} is a finite dimensional simple Lie algebra. For $n \in \mathbb{Z}_{>0}$, let $\sigma = (\sigma_1, \dots, \sigma_n) \in \text{Aut}_{\text{cfo}}^n(\mathfrak{g})$ and $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{>0}^n$ satisfying $\sigma^{\mathbf{m}} = \text{id}$. In the following, we define a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}^\sigma$ and an abelian group $Q_{\mathfrak{h}}$, and then we define a *multiloop Lie algebra* $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ as a $Q_{\mathfrak{h}} \times \mathbb{Z}^n$ -graded Lie algebra.

First, we assume that $\mathfrak{g}^\sigma \neq \{0\}$. In this case, we take \mathfrak{h} as a Cartan subalgebra of \mathfrak{g}^σ . By Lemma 3.1.3 (b), we can define the root space decomposition of \mathfrak{g} with respect to \mathfrak{h} , which we denote by $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha := \{g \in \mathfrak{g} \mid [h, g] = \langle \alpha, h \rangle g \text{ for } h \in \mathfrak{h}\}$. Put

$$\Delta = \text{supp}_{\mathfrak{h}^*}(\mathfrak{g}) \setminus \{0\} \subseteq \mathfrak{h}^*,$$

and let $Q_{\mathfrak{h}} := \sum_{\alpha \in \Delta} \mathbb{Z}\alpha \subseteq \mathfrak{h}^*$. This grading, together with the grading defined in (9), gives a $Q_{\mathfrak{h}} \times \bar{\Lambda}_{\mathbf{m}}$ -grading on \mathfrak{g} as

$$\mathfrak{g} = \bigoplus_{(\alpha, \bar{\lambda}) \in Q_{\mathfrak{h}} \times \bar{\Lambda}_{\mathbf{m}}} \mathfrak{g}_\alpha^{\bar{\lambda}}. \quad (15)$$

Then we can define a $Q_{\mathfrak{h}} \times \mathbb{Z}^n$ -graded Lie algebra $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ as

$$L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h}) = \bigoplus_{(\alpha, \lambda) \in Q_{\mathfrak{h}} \times \mathbb{Z}^n} \mathfrak{g}_\alpha^{\bar{\lambda}} \otimes t^\lambda.$$

Next, we assume that $\mathfrak{g}^\sigma = \{0\}$. For the notational convenience, in this case we let $\mathfrak{h} = \mathfrak{g}^\sigma = \{0\}$ and $Q_{\mathfrak{h}}$ be a trivial group, and we define

$$L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h}) = \bigoplus_{\lambda \in \mathbb{Z}^n} \mathfrak{g}^{\bar{\lambda}} \otimes t^\lambda.$$

Also in this case, we consider $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ as a $Q_{\mathfrak{h}} \times \mathbb{Z}^n (\cong \mathbb{Z}^n)$ -graded Lie algebra. Note that, as a \mathbb{Z}^n -graded Lie algebra, $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h}) = M_{\mathbf{m}}(\mathfrak{g}, \sigma)$.

Definition 3.2.1. Suppose that \mathfrak{g} is a finite dimensional simple Lie algebra, $\sigma \in \text{Aut}_{\text{cfo}}^n(\mathfrak{g})$, and $\mathbf{m} \in \mathbb{Z}_{>0}^n$ such that $\sigma^{\mathbf{m}} = \text{id}$. Then we call the $Q_{\mathfrak{h}} \times \mathbb{Z}^n$ -graded Lie algebra $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ defined above the *multiloop Lie algebra* determined by $\mathfrak{g}, \sigma, \mathbf{m}, \mathfrak{h}$. We call the positive number n the *nullity* of $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$.

Remark 3.2.2. (a) In the definition of a multiloop algebra $M_{\mathbf{m}}(\mathcal{A}, \sigma)$, \mathcal{A} is not supposed to be either finite dimensional or simple. Thus, it may be more appropriate to call $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ in Definition 3.2.1 a multiloop Lie algebra based on a finite dimensional simple Lie algebra. In this paper, however, we consider a finite dimensional simple case only. Thus, we call it simply multiloop Lie algebra.

(b) Even in the case where $\mathfrak{g}^{\sigma} \neq \{0\}$, $\Delta := \text{supp}_{Q_{\mathfrak{h}}}(\mathfrak{g}) \setminus \{0\}$ does not necessarily coincide with the root system of \mathfrak{g} since \mathfrak{h} is not necessarily a Cartan subalgebra of \mathfrak{g} . It is, however, proved in the next subsection that Δ is a irreducible (possibly non-reduced) finite root system.

Henceforth, we identify \mathfrak{g}^{σ} with $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})^0$ and consider \mathfrak{g}^{σ} as a Lie subalgebra of $L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ by this identification.

3.3. The property of Δ . Let \mathfrak{g} be a finite dimensional simple Lie algebra, $\sigma = (\sigma_1, \dots, \sigma_n) \in \text{Aut}_{\text{cfo}}^n(\mathfrak{g})$, $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{>0}^n$ where $\sigma^{\mathbf{m}} = \text{id}$, and suppose that $\mathfrak{g}^{\sigma} \neq \{0\}$. We take a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}^{\sigma}$, and define a $Q_{\mathfrak{h}} \times \bar{\Lambda}_{\mathbf{m}}$ -grading on \mathfrak{g} as (15). Put $\Delta = \text{supp}_{Q_{\mathfrak{h}}}(\mathfrak{g})$.

First, since \mathfrak{g} is finite dimensional, the following lemma is obvious:

Lemma 3.3.1. Δ is a finite set. □

Next, by Lemma 3.1.3 (a), we can define an isomorphism $\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$ canonically by setting

$$\langle \nu(h), h_1 \rangle = (h|h_1) \quad \text{for } h, h_1 \in \mathfrak{h}.$$

Then we can also define a non-degenerate bilinear form (\mid) on \mathfrak{h}^* by setting

$$(\alpha|\beta) = (\nu^{-1}(\alpha)|\nu^{-1}(\beta)) \quad \text{for } \alpha, \beta \in \mathfrak{h}^*. \quad (16)$$

Lemma 3.3.2. The k -span of Δ coincides with \mathfrak{h}^* .

Proof. We assume that the k -span of Δ does not coincide with \mathfrak{h}^* . Then there exists some non-zero element $h \in \mathfrak{h}$ such that $\langle \alpha, h \rangle = 0$ for all $\alpha \in \Delta$, which means that $[h, \mathfrak{g}_{\alpha}] = 0$ for all $\alpha \in \Delta$. Hence, we have $[h, \mathfrak{g}] = 0$ since $h \in \mathfrak{h}$, which contradicts the simplicity of \mathfrak{g} . □

Let $\alpha \in \Delta$ and $\bar{\lambda} \in \bar{\Lambda}_{\mathbf{m}}$ such that $\mathfrak{g}_{\alpha}^{\bar{\lambda}} \neq \{0\}$. We have using (10) and (11) that $\mathfrak{g}_{-\alpha}^{-\bar{\lambda}} \neq \{0\}$ since (\mid) is non-degenerate on \mathfrak{g} . Thus, we can take non-zero elements $x_{\alpha}^{\bar{\lambda}} \in \mathfrak{g}_{\alpha}^{\bar{\lambda}}$ and $x_{-\alpha}^{-\bar{\lambda}} \in \mathfrak{g}_{-\alpha}^{-\bar{\lambda}}$. For $h \in \mathfrak{h}$, we have

$$(h|x_{\alpha}^{\bar{\lambda}}, x_{-\alpha}^{-\bar{\lambda}}) = ([h, x_{\alpha}^{\bar{\lambda}}]|x_{-\alpha}^{-\bar{\lambda}}) = \langle \alpha, h \rangle (x_{\alpha}^{\bar{\lambda}}|x_{-\alpha}^{-\bar{\lambda}}) = (h|\nu^{-1}(\alpha))(x_{\alpha}^{\bar{\lambda}}|x_{-\alpha}^{-\bar{\lambda}}).$$

Thus we have

$$[x_{\alpha}^{\bar{\lambda}}, x_{-\alpha}^{-\bar{\lambda}}] = (x_{\alpha}^{\bar{\lambda}}|x_{-\alpha}^{-\bar{\lambda}})\nu^{-1}(\alpha) \in \mathfrak{h} \quad (17)$$

since (\mid) is non-degenerate on \mathfrak{h} .

Lemma 3.3.3. For $\alpha \in \Delta$, $(\alpha|\alpha) \neq 0$.

Proof. For some $\alpha \in \Delta$, we assume that $\langle \alpha | \alpha \rangle = \langle \alpha, \nu^{-1}(\alpha) \rangle = 0$. We can take $0 \neq x_\alpha^\lambda \in \mathfrak{g}_\alpha^\lambda$ for some $\bar{\lambda} \in \bar{\Lambda}_m$. Then there exists some element $x_{-\alpha}^{-\bar{\lambda}} \in \mathfrak{g}_{-\alpha}^{-\bar{\lambda}}$ such that $(x_\alpha^\lambda | x_{-\alpha}^{-\bar{\lambda}}) = 1$. By (17) and the assumption, we can see that the Lie subalgebra of \mathfrak{g} spanned by $\{\nu^{-1}(\alpha), x_\alpha^\lambda, x_{-\alpha}^{-\bar{\lambda}}\}$, which we denote by S , is a three-dimensional nilpotent Lie algebra. Then since $\text{ad}_{\mathfrak{g}}(S) \simeq S$ is also nilpotent (in particular, solvable) and $\text{ad}_{\mathfrak{g}}(\nu^{-1}(\alpha)) \in [\text{ad}_{\mathfrak{g}}(S), \text{ad}_{\mathfrak{g}}(S)]$, it follows from the Lie's theorem that $\text{ad}_{\mathfrak{g}}(\nu^{-1}(\alpha))$ acts nilpotently on \mathfrak{g} . From this and Lemma 3.1.3 (b), it follows that $\text{ad}_{\mathfrak{g}}(\nu^{-1}(\alpha)) = 0$. This forces $\alpha = 0$, and this is contradiction since $0 \notin \Delta$. \square

Let $\alpha \in \Delta$ and $\bar{\lambda} \in \bar{\Lambda}_m$ such that $\mathfrak{g}_\alpha^\lambda \neq \{0\}$. By Lemma 3.3.3, $2(\alpha | \alpha)^{-1} \in k$ exists. Thus, we can choose non-zero elements $x_\alpha^\lambda \in \mathfrak{g}_\alpha^\lambda$ and $x_{-\alpha}^{-\bar{\lambda}} \in \mathfrak{g}_{-\alpha}^{-\bar{\lambda}}$ satisfying

$$(x_\alpha^\lambda | x_{-\alpha}^{-\bar{\lambda}}) = \frac{2}{(\alpha | \alpha)},$$

and we set

$$h_\alpha = \frac{2\nu^{-1}(\alpha)}{(\alpha | \alpha)} \in \mathfrak{h}. \quad (18)$$

Then we have

$$[h_\alpha, x_\alpha^\lambda] = 2x_\alpha^\lambda, \quad [h_\alpha, x_{-\alpha}^{-\bar{\lambda}}] = -2x_{-\alpha}^{-\bar{\lambda}}, \quad (19)$$

and using (17),

$$[x_\alpha^\lambda, x_{-\alpha}^{-\bar{\lambda}}] = h_\alpha. \quad (20)$$

By (19) and (20), we can see that the Lie subalgebra of \mathfrak{g} spanned by these three elements $\{x_\alpha^\lambda, x_{-\alpha}^{-\bar{\lambda}}, h_\alpha\}$ is isomorphic to $\mathfrak{sl}_2(k)$. We call the set of these three elements a $\mathfrak{sl}_2(k)$ -triple with respect to $(\alpha, \bar{\lambda})$. Note that this set is defined only for the pair $(\alpha, \bar{\lambda})$ satisfying $\mathfrak{g}_\alpha^\lambda \neq \{0\}$. Also, note that for some $\alpha \in \Delta$ it is possible that h_α is contained in more than one $\mathfrak{sl}_2(k)$ -triples.

For $\alpha \in \Delta$, we define a reflection s_α on \mathfrak{h}^* by

$$s_\alpha(\gamma) = \gamma - \langle \gamma, h_\alpha \rangle \alpha \text{ for } \gamma \in \mathfrak{h}^*. \quad (21)$$

Lemma 3.3.4. *Let $\alpha, \beta \in \Delta$, then*

- (a) $\langle \beta, h_\alpha \rangle \in \mathbb{Z}$,
- (b) $s_\alpha(\Delta) = \Delta$.

Proof. We have some $\bar{\lambda}, \bar{\mu} \in \bar{\Lambda}_m$ such that $\mathfrak{g}_\alpha^\lambda \neq \{0\}$ and $\mathfrak{g}_\beta^{\bar{\mu}} \neq \{0\}$, and by the above construction we can take a $\mathfrak{sl}_2(k)$ -triple $\{x_\alpha^\lambda, x_{-\alpha}^{-\bar{\lambda}}, h_\alpha\}$ with respect to $(\alpha, \bar{\lambda})$. Let S_α^λ be the subalgebra of \mathfrak{g} spanned by these elements.

(a) We can consider \mathfrak{g} as a S_α^λ -module by the adjoint action. Since $\mathfrak{g}_\beta^{\bar{\mu}}$ is nonzero eigenspace for h_α , (a) follows from the representation theory of $\mathfrak{sl}_2(k)$.

(b) It suffices to show that

$$s_\alpha(\beta) \in \Delta. \quad (22)$$

We construct an automorphism of \mathfrak{g} using the elements x_α^λ and $x_{-\alpha}^{-\bar{\lambda}}$. Since Δ is a finite set and

$$\text{ad}(x_\alpha^\lambda)(\mathfrak{g}_\gamma) \subseteq \mathfrak{g}_{\alpha+\gamma}$$

for $\gamma \in \Delta \cup \{0\}$, we can see that $\text{ad}(x_\alpha^\lambda)$ is nilpotent, and so is $\text{ad}(x_{-\alpha}^{-\bar{\lambda}})$. Therefore,

$$\theta_\alpha^{\bar{\lambda}} := \exp(\text{ad}(x_\alpha^\lambda)) \exp(-\text{ad}(x_{-\alpha}^{-\bar{\lambda}})) \exp(\text{ad}(x_\alpha^\lambda)) \in \text{Aut}(\mathfrak{g})$$

is a well-defined automorphism of \mathfrak{g} . To show (22), it suffices to show that

$$\theta_\alpha^\lambda(\mathfrak{g}_\beta) \subseteq \mathfrak{g}_{s_\alpha(\beta)}.$$

Let $x_\beta \in \mathfrak{g}_\beta$. For $h \in \mathfrak{h}$ such that $\langle \alpha, h \rangle = 0$, using $\theta_\alpha^\lambda(h) = h$, we have

$$[h, \theta_\alpha^\lambda(x_\beta)] = \theta_\alpha^\lambda([h, x_\beta]) = \langle \beta, h \rangle \theta_\alpha^\lambda(x_\beta) = \langle s_\alpha(\beta), h \rangle \theta_\alpha^\lambda(x_\beta).$$

Thus, we have only to check that

$$[h_\alpha, \theta_\alpha^\lambda(x_\beta)] = \langle s_\alpha(\beta), h_\alpha \rangle \theta_\alpha^\lambda(x_\beta).$$

This follows from

$$\theta_\alpha^\lambda(h_\alpha) = -h_\alpha \quad (23)$$

and

$$\langle s_\alpha(\beta), h_\alpha \rangle = \langle \beta - \langle \beta, h_\alpha \rangle \alpha, h_\alpha \rangle = -\langle \beta, h_\alpha \rangle$$

((23) follows from an easy calculation in $\mathfrak{sl}_2(k)$). \square

Now, we show the following proposition:

Proposition 3.3.5. *Let \mathfrak{g} be a finite dimensional simple Lie algebra and $\sigma = (\sigma_1, \dots, \sigma_n) \in \text{Aut}_{\text{cfo}}^n(\mathfrak{g})$ such that $\mathfrak{g}^\sigma \neq \{0\}$, and let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g}^σ . Then $\Delta := \text{supp}_{Q_\mathfrak{h}}(\mathfrak{g}) \setminus \{0\}$ is an irreducible (possibly non-reduced) finite root system (cf. [B2, Chapter IV]).*

Proof. By Lemma 3.3.1, 3.3.2, (18), (21), Lemma 3.3.4 (a) and (b), we have that Δ is a (possibly non-reduced) finite root system. Thus, it suffices to show that Δ is irreducible. We assume that $\Delta = \Delta_1 \cup \Delta_2$, $(\Delta_1 | \Delta_2) = 0$ and $\Delta_1 \neq \emptyset$. Let $\mathfrak{g}(\Delta_1)$ be a subalgebra in \mathfrak{g} generated by $\cup_{\alpha \in \Delta_1} \mathfrak{g}_\alpha$. If $\alpha \in \Delta_1$, $\beta \in \Delta_2$, we have from Lemma 3.3.3 that $(\alpha + \beta | \alpha) \neq 0$ $(\alpha + \beta | \beta) \neq 0$, and then we have $\alpha + \beta \notin \Delta$. Thus, since $\alpha + \beta \neq 0$ we have $\mathfrak{g}_{\alpha+\beta} = \{0\}$, and this means

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0. \quad (24)$$

Then we can easily see that $\mathfrak{g}(\Delta_1)$ is a nonzero ideal of \mathfrak{g} , which coincides with \mathfrak{g} . Since $[\mathfrak{g}_\beta, \mathfrak{g}(\Delta_1)] = 0$ for any $\beta \in \Delta_2$ by (24), $\Delta_2 = \emptyset$. \square

Then the following corollary is obvious from the definition of a multiloop Lie algebra $L_\mathbf{m}(\mathfrak{g}, \sigma, \mathfrak{h})$.

Corollary 3.3.6. *Let $\mathfrak{g}, \mathfrak{h}, \sigma$ be as in Proposition 3.3.5 (in particular, $\mathfrak{g}^\sigma \neq \{0\}$), and let $\mathbf{m} \in \mathbb{Z}^n$ satisfy $\sigma^\mathbf{m} = \text{id}$. Then $\Delta := \text{supp}_{Q_\mathfrak{h}}(L_\mathbf{m}(\mathfrak{g}, \sigma, \mathfrak{h})) \setminus \{0\}$ is an irreducible (possibly non-reduced) finite root system.* \square

4. SUPPORT-ISOMORPHISM OF MULTILOOP LIE ALGEBRAS

Let $\mathcal{L} = L_\mathbf{m}(\mathfrak{g}, \sigma, \mathfrak{h})$, $\mathcal{L}' = L_{\mathbf{m}'}(\mathfrak{g}', \sigma', \mathfrak{h}')$ be multiloop Lie algebras of nullity n . As defined in the previous section, \mathcal{L} is $Q_\mathfrak{h} \times \mathbb{Z}^n$ -graded and \mathcal{L}' is $Q_{\mathfrak{h}'} \times \mathbb{Z}^n$ -graded. Thus, \mathcal{L} and \mathcal{L}' are support-isomorphic if and only if there exist a Lie algebra isomorphism $\varphi : \mathcal{L} \rightarrow \mathcal{L}'$ and a group isomorphism $\varphi_{\text{su}} : \langle \text{supp}_{Q_\mathfrak{h} \times \mathbb{Z}^n}(\mathcal{L}) \rangle \rightarrow \langle \text{supp}_{Q_{\mathfrak{h}'} \times \mathbb{Z}^n}(\mathcal{L}') \rangle$ such that

$$\varphi(\mathcal{L}_\alpha^\lambda) = \mathcal{L}'_{\alpha'}^{\lambda'}$$

for $(\alpha, \lambda) \in \langle \text{supp}_{Q_\mathfrak{h} \times \mathbb{Z}^n}(\mathcal{L}) \rangle$ where we set $\varphi_{\text{su}}((\alpha, \lambda)) = (\alpha', \lambda')$. The goal of this section is to give a necessary and sufficient condition for \mathcal{L} and \mathcal{L}' to be support-isomorphic.

4.1. Some isomorphisms. In section 2, we have observed the conditions for two multiloop algebras, which are \mathbb{Z}^n -graded, to be isograded-isomorphic or support-isomorphic. To apply those results to multiloop Lie algebras, which are $Q_{\mathfrak{h}} \times \mathbb{Z}^n$ -graded, we define the following:

Definition 4.1.1. Let \mathfrak{L} and \mathfrak{L}' be multiloop Lie algebras of nullity n . Note that we can see \mathfrak{L} and \mathfrak{L}' as \mathbb{Z}^n -graded Lie algebras by considering only their \mathbb{Z}^n -gradings.

- (a) We say \mathfrak{L} and \mathfrak{L}' are \mathbb{Z}^n -isograded-isomorphic if \mathfrak{L} and \mathfrak{L}' are isograded-isomorphic as \mathbb{Z}^n -graded Lie algebras. In that case we write $\mathfrak{L} \cong_{\mathbb{Z}^n\text{-ig}} \mathfrak{L}'$.
- (b) We say \mathfrak{L} and \mathfrak{L}' are \mathbb{Z}^n -support-isomorphic if \mathfrak{L} and \mathfrak{L}' are support-isomorphic as \mathbb{Z}^n -graded Lie algebras. In that case we write $\mathfrak{L} \cong_{\mathbb{Z}^n\text{-su}} \mathfrak{L}'$.

The following lemma immediately follows from Proposition 2.2.5:

Lemma 4.1.2. Let $\mathfrak{L} = L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ and $\mathfrak{L}' = L_{\mathbf{m}'}(\mathfrak{g}', \sigma', \mathfrak{h}')$ be multiloop Lie algebras of nullity n . Then $\mathfrak{L} \cong_{\mathbb{Z}^n\text{-su}} \mathfrak{L}'$ if and only if there exist $P \in \text{GL}_n(\mathbb{Z})$ and an algebra isomorphism $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}'$ such that $\sigma' = \varphi \sigma^P \varphi^{-1}$. \square

The following proposition, which can be proved in the almost same way used in the proof of [ABFP2, Proposition 2.1.3], shows that if two multiloop Lie algebras are \mathbb{Z}^n -isograded-isomorphic or \mathbb{Z}^n -support-isomorphic, then we can choose the isomorphism preserving the root grading. In particular, if two multiloop Lie algebras are \mathbb{Z}^n -isograded-isomorphic (resp. \mathbb{Z}^n -support-isomorphic), then they are isograded-isomorphic (resp. support-isomorphic).

Proposition 4.1.3. Let $\mathfrak{L} = L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ and $\mathfrak{L}' = L_{\mathbf{m}'}(\mathfrak{g}', \sigma', \mathfrak{h}')$ be multiloop Lie algebras of nullity n . If \mathfrak{L} and \mathfrak{L}' are \mathbb{Z}^n -isograded-isomorphic (resp. \mathbb{Z}^n -support-isomorphic), then we can choose a \mathbb{Z}^n -isograded-isomorphism (resp. \mathbb{Z}^n -support-isomorphism) φ satisfying the following condition: there exists a group isomorphism $\varphi_Q : Q_{\mathfrak{h}} \rightarrow Q_{\mathfrak{h}'}$ satisfying

$$\varphi(\mathfrak{L}_{\alpha}) = \mathfrak{L}'_{\varphi_Q(\alpha)} \quad (25)$$

for $\alpha \in Q_{\mathfrak{h}}$.

Proof. We only show the \mathbb{Z}^n -isograded-isomorphic case. (The proof of the other case is same). Let $\varphi' : \mathfrak{L} \rightarrow \mathfrak{L}'$ be a \mathbb{Z}^n -isograded-isomorphism. If $\mathfrak{g}^{\sigma} = \{0\}$, we have $\mathfrak{g}'^{\sigma'} = \mathfrak{L}'^0 = \varphi'(\mathfrak{L}^0) = \{0\}$. In this case both $Q_{\mathfrak{h}}$ and $Q_{\mathfrak{h}'}$ being trivial groups, (25) obviously follows if we put $\varphi = \varphi'$. Next, suppose $\mathfrak{g}^{\sigma} \neq \{0\}$. If a \mathbb{Z}^n -isograded-automorphism $\psi : \mathfrak{L} \rightarrow \mathfrak{L}$ satisfies $\varphi' \circ \psi(\mathfrak{h}) = \mathfrak{h}'$, then it is easily checked that $\varphi = \varphi' \circ \psi$ satisfies (25) for suitable φ_Q . Thus, we show that there exists ψ satisfying the above condition. By Lemma 3.1.1, we can write

$$\mathfrak{g}^{\sigma} = \mathfrak{s}_0 \oplus \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k$$

where \mathfrak{s}_0 is a center and \mathfrak{s}_i for $1 \leq i \leq k$ is a simple ideal. Also, since \mathfrak{h} and $\varphi'^{-1}(\mathfrak{h}')$ are both the Cartan subalgebras of \mathfrak{g}^{σ} , we can write

$$\mathfrak{h} = \mathfrak{s}_0 \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k \text{ and } \varphi'^{-1}(\mathfrak{h}') = \mathfrak{s}_0 \oplus \mathfrak{h}'_1 \oplus \cdots \oplus \mathfrak{h}'_k$$

where $\mathfrak{h}_i, \mathfrak{h}'_i$ are both the Cartan subalgebras of \mathfrak{s}_i . Using the technique in the proof of [ABFP2, Proposition 2.1.3], we can take \mathbb{Z}^n -isograded-automorphisms ψ_i of \mathfrak{L} for $1 \leq i \leq k$ such that $\psi_i(\mathfrak{h}_i) = \mathfrak{h}'_i$ and $\psi_i(g) = g$ for $g \in \mathfrak{s}_j$ if $i \neq j$. Then $\psi := \psi_1 \circ \cdots \circ \psi_k$ satisfies $\varphi' \circ \psi(\mathfrak{h}) = \mathfrak{h}'$. \square

4.2. Support-isomorphism of multiloop Lie algebras.

Definition 4.2.1. Suppose that Q, Λ are abelian groups, and \mathfrak{B} is a $Q \times \Lambda$ -graded Lie algebra.

(a) Let $\rho : \langle \text{supp}_\Lambda(\mathfrak{B}) \rangle \rightarrow \Lambda$ be a group monomorphism. We define a new $Q \times \Lambda$ -graded Lie algebra $\mathfrak{B}_{(\rho)}$ as follows: $\mathfrak{B}_{(\rho)} = \mathfrak{B}$ as a Lie algebra, and the $Q \times \Lambda$ -grading on $\mathfrak{B}_{(\rho)}$ is given by

$$(\mathfrak{B}_{(\rho)})_\alpha^\lambda = \begin{cases} \mathfrak{B}_\alpha^{\rho^{-1}(\lambda)} & \text{if } \lambda \in \text{Im } \rho \\ \{0\} & \text{if } \lambda \notin \text{Im } \rho \end{cases} \quad (26)$$

for $\alpha \in Q, \lambda \in \Lambda$.

(b) Let $s \in \text{Hom}(Q, \Lambda)$ be a group homomorphism from Q to Λ . We define a new $Q \times \Lambda$ -graded Lie algebra $\mathfrak{B}^{(s)}$ as follows: as a Lie algebra, $\mathfrak{B}^{(s)} = \mathfrak{B}$ and the grading on $\mathfrak{B}^{(s)}$ is given by

$$(\mathfrak{B}^{(s)})_\alpha^\lambda = \mathfrak{B}_\alpha^{\lambda+s(\alpha)}$$

for $\alpha \in Q, \lambda \in \Lambda$.

Remark 4.2.2. Under the notation of Definition 4.2.1 (a), we have $\text{supp}_\Lambda \mathfrak{B}_{(\rho)} = \rho(\text{supp}_\Lambda(\mathfrak{B}))$. Thus, we have

$$\langle \text{supp}_\Lambda(\mathfrak{B}_{(\rho)}) \rangle = \rho(\langle \text{supp}_\Lambda(\mathfrak{B}) \rangle). \quad (27)$$

Lemma 4.2.3. Suppose that $\mathfrak{L} = L_{\tilde{\mathbf{m}}}(\mathfrak{g}, \sigma, \mathfrak{h})$ is a multiloop Lie algebra, and suppose that $P \in \text{GL}_n(\mathbb{Z}), \tilde{\mathbf{m}} \in \mathbb{Z}_{>0}^n$ satisfy $(\sigma^P)^{\tilde{\mathbf{m}}} = \text{id}$. Then $\mathfrak{g}^\sigma = \mathfrak{g}^{\sigma^P}$, and there exists some group monomorphism $\rho : \langle \text{supp}_{\mathbb{Z}^n}(\mathfrak{L}) \rangle \rightarrow \mathbb{Z}^n$ such that $\mathfrak{L}_{(\rho)}$ is $Q_{\mathfrak{h}} \times \mathbb{Z}^n$ -graded isomorphic to $L_{\tilde{\mathbf{m}}}(\mathfrak{g}, \sigma^P, \mathfrak{h})$.

Proof. By the definition of $\sigma^P, \mathfrak{g}^\sigma \subseteq \mathfrak{g}^{\sigma^P}$ is obvious. Then, since $(\sigma^P)^{P^{-1}} = \sigma$, we have $\mathfrak{g}^\sigma = \mathfrak{g}^{\sigma^P}$. We write $\mathfrak{L}' = L_{\tilde{\mathbf{m}}}(\mathfrak{g}, \sigma^P, \mathfrak{h})$. By Proposition 2.2.5, we can take a \mathbb{Z}^n -support-isomorphism $\psi : \mathfrak{L} \rightarrow \mathfrak{L}'$ such that $\psi|_{\mathfrak{g}^\sigma} = \text{id}_{\mathfrak{g}^\sigma}$. Then, since $\psi|_{\mathfrak{h}} = \text{id}_{\mathfrak{h}}$, it is easily checked that

$$\psi(\mathfrak{L}_\alpha) = \mathfrak{L}'_\alpha$$

for $\alpha \in Q_{\mathfrak{h}}$. Let $\psi_{\text{su}} : \langle \text{supp}_{\mathbb{Z}^n}(\mathfrak{L}) \rangle \rightarrow \langle \text{supp}_{\mathbb{Z}^n}(\mathfrak{L}') \rangle$ be a group isomorphism such that $\psi(\mathfrak{L}^\lambda) = \mathfrak{L}'^{\psi_{\text{su}}(\lambda)}$ for $\lambda \in \langle \text{supp}_{\mathbb{Z}^n}(\mathfrak{L}) \rangle$, and $\iota : \langle \text{supp}_{\mathbb{Z}^n}(\mathfrak{L}') \rangle \rightarrow \mathbb{Z}^n$ be a canonical monomorphism. We show that $\mathfrak{L}_{(\iota \circ \psi_{\text{su}})}$ is $Q_{\mathfrak{h}} \times \mathbb{Z}^n$ -graded isomorphic to \mathfrak{L}' . Since $\mathfrak{L}_{(\iota \circ \psi_{\text{su}})} = \mathfrak{L}$ as a Lie algebra, we can see ψ as a Lie algebra isomorphism from $\mathfrak{L}_{(\iota \circ \psi_{\text{su}})}$ onto \mathfrak{L}' . If $\lambda \in \langle \text{supp}_{\mathbb{Z}^n}(\mathfrak{L}') \rangle$, then

$$\psi \left((\mathfrak{L}_{(\iota \circ \psi_{\text{su}})})_\alpha^\lambda \right) = \psi \left(\mathfrak{L}_\alpha^{\psi_{\text{su}}^{-1}(\lambda)} \right) = \mathfrak{L}'_\alpha^\lambda$$

for $\alpha \in Q_{\mathfrak{h}}$. Also if $\lambda \notin \langle \text{supp}_{\mathbb{Z}^n}(\mathfrak{L}') \rangle$,

$$\psi \left((\mathfrak{L}_{(\iota \circ \psi_{\text{su}})})_\alpha^\lambda \right) = \{0\} = \mathfrak{L}'_\alpha^\lambda$$

for $\alpha \in Q_{\mathfrak{h}}$. Thus, ψ is indeed a $Q_{\mathfrak{h}} \times \mathbb{Z}^n$ -graded-isomorphism. \square

For an algebra \mathcal{A} and $\tau = (\tau_1, \dots, \tau_n), \sigma = (\sigma_1, \dots, \sigma_n) \in \text{Aut}(\mathcal{A})^n$, we write $\tau\sigma = (\tau_1\sigma_1, \dots, \tau_n\sigma_n) \in \text{Aut}(\mathcal{A})^n$.

Lemma 4.2.4. *Let $\mathfrak{L} = L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ be a multiloop Lie algebra of nullity n such that $\mathfrak{g}^\sigma \neq \{0\}$, and let $s = (s_1, \dots, s_n) \in \text{Hom}(Q_{\mathfrak{h}}, \mathbb{Z}^n)$. For $1 \leq i \leq n$, we define $\tau_i \in \text{Aut}(\mathfrak{g})$ by*

$$\tau_i(x_\alpha) = \zeta_{m_i}^{-s_i(\alpha)}(x_\alpha)$$

for $\alpha \in Q_{\mathfrak{h}}, x_\alpha \in \mathfrak{g}_\alpha$. Then

$$\tau\sigma \in \text{Aut}_{\text{cfo}}^n(\mathfrak{g}), (\tau\sigma)^m = \text{id}, \quad (28)$$

and \mathfrak{h} is a Cartan subalgebra of $\mathfrak{g}^{\tau\sigma}$. Moreover, $\mathfrak{L}^{(s)}$ is $Q_{\mathfrak{h}} \times \mathbb{Z}^n$ -graded-isomorphic to $L_{\mathbf{m}}(\mathfrak{g}, \tau\sigma, \mathfrak{h})$.

Proof. It is clear that $\sigma_1, \dots, \sigma_n, \tau_1, \dots, \tau_n$ commute with each other and $\tau^m = \text{id}$. Then (28) is easily checked. $\mathfrak{h} \subseteq \mathfrak{g}^{\tau\sigma}$ is obvious. If $g \in \mathfrak{g}^{\tau\sigma}$ satisfies $[\mathfrak{h}, g] = 0$, that is $g \in \mathfrak{g}^{\tau\sigma} \cap \mathfrak{g}_0$, then we have using $\tau_i|_{\mathfrak{g}_0} = \text{id}$ for all i that $g \in \mathfrak{g}^{\tau\sigma} \cap \mathfrak{g}_0 = \mathfrak{g}^\sigma \cap \mathfrak{g}_0 = \mathfrak{h}$. Therefore, \mathfrak{h} is a Cartan subalgebra of $\mathfrak{g}^{\tau\sigma}$. For $\alpha \in Q_{\mathfrak{h}}, \lambda = (l_1, \dots, l_n) \in \mathbb{Z}^n$,

$$\begin{aligned} \mathfrak{g}_\alpha^{\bar{\lambda}(\tau\sigma, m)} &= \{g \in \mathfrak{g}_\alpha \mid \tau_i \sigma_i(g) = \zeta_{m_i}^{l_i} g \text{ for } 1 \leq i \leq n\} \\ &= \{g \in \mathfrak{g}_\alpha \mid \sigma_i(g) = \zeta_{m_i}^{l_i + s_i(\alpha)} g \text{ for } 1 \leq i \leq n\} \\ &= \mathfrak{g}_\alpha^{\overline{\lambda + s(\alpha)}(\sigma, m)}. \end{aligned}$$

Therefore,

$$L_{\mathbf{m}}(\mathfrak{g}, \tau\sigma, \mathfrak{h})_\alpha^\lambda \ni g \otimes t^\lambda \mapsto g \otimes t^{\lambda + s(\alpha)} \in (\mathfrak{L}^{(s)})_\alpha^\lambda = \mathfrak{L}_\alpha^{\lambda + s(\alpha)}$$

defines a $Q_{\mathfrak{h}} \times \mathbb{Z}^n$ -graded-isomorphism. \square

We introduce the following notation: if $q \in \mathbb{Q}$ is expressed as $q = a/b$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{>0}$, then we set

$$\zeta^q = \zeta_b^a.$$

By (1), ζ^q is well-defined.

Now, we show the following theorem:

Theorem 4.2.5. *Let $\mathfrak{L} = L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ and $\mathfrak{L}' = L_{\mathbf{m}'}(\mathfrak{g}', \sigma', \mathfrak{h}')$ be multiloop Lie algebras of nullity n . Then the following statements are equivalent:*

- (a) $\mathfrak{L} \cong_{\text{supp}} \mathfrak{L}'$.
- (b) *There exist $s = (s_1, \dots, s_n) \in \text{Hom}(Q_{\mathfrak{h}}, \mathbb{Q}^n)$, $P \in \text{GL}_n(\mathbb{Z})$ and a Lie algebra isomorphism $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}'$ satisfying the following condition: if we define $\tau_i \in \text{Aut}(\mathfrak{g})$ for $1 \leq i \leq n$ as*

$$\tau_i(x_\alpha) = \zeta^{-s_i(\alpha)} x_\alpha \quad (29)$$

for $\alpha \in Q_{\mathfrak{h}}, x_\alpha \in \mathfrak{g}_\alpha$ and $\tau = (\tau_1, \dots, \tau_n)$, then

$$\sigma' = \varphi(\tau\sigma)^P \varphi^{-1}.$$

- (c) *There exists a finite sequence of $Q_{\mathfrak{h}} \times \mathbb{Z}^n$ -graded Lie algebras $\mathfrak{L}_0, \mathfrak{L}_1, \dots, \mathfrak{L}_p$ satisfying the following three conditions:*
 - (i) $\mathfrak{L}_0 = \mathfrak{L}$.
 - (ii) $\mathfrak{L}_p \cong_{\mathbb{Z}^n\text{-ig}} \mathfrak{L}'$.
 - (iii) *For $1 \leq i \leq p-1$, \mathfrak{L}_{i+1} is either $\mathfrak{L}_{i(\rho_i)}$ for some group monomorphism $\rho_i : \langle \text{supp}_{\mathbb{Z}^n}(\mathfrak{L}_i) \rangle \rightarrow \mathbb{Z}^n$ or $\mathfrak{L}_i^{(s_i)}$ for some $s_i \in \text{Hom}(Q_{\mathfrak{h}}, \mathbb{Z}^n)$.*

Proof. “(a) \Rightarrow (b)” If $\mathfrak{g}^\sigma = \{0\}$, then $\mathfrak{L} \cong_{\text{supp}} \mathfrak{L}'$ means $\mathfrak{L} \cong_{\mathbb{Z}^n - \text{su}} \mathfrak{L}'$, and (b) follows from Lemma 4.1.2. Thus, we suppose that $\mathfrak{g}^\sigma \neq \{0\}$, and let $\psi : \mathfrak{L} \rightarrow \mathfrak{L}'$ be a support-isomorphism. Then $\psi(\mathfrak{h}) = \mathfrak{h}'$, and if we define $\hat{\psi} : \mathfrak{h}^* \rightarrow \mathfrak{h}'^*$ as $\langle \hat{\psi}(\alpha), \psi(h) \rangle = \langle \alpha, h \rangle$ for $\alpha \in \mathfrak{h}^*, h \in \mathfrak{h}$, it is easily checked that $\psi(\mathfrak{L}_\alpha) = \mathfrak{L}'_{\hat{\psi}(\alpha)}$ for $\alpha \in Q_{\mathfrak{h}}$. Thus, we can see $\hat{\psi}$ as a group isomorphism from $Q_{\mathfrak{h}}$ onto $Q_{\mathfrak{h}'}$. We define a group homomorphism $p : \langle \text{supp}_{Q_{\mathfrak{h}} \times \mathbb{Z}^n}(\mathfrak{L}) \rangle \rightarrow \mathbb{Z}^n$ as

$$\psi(\mathfrak{L}_\alpha^\lambda) = \mathfrak{L}'_{\hat{\psi}(\alpha)}^{p(\alpha, \lambda)}$$

for $(\alpha, \lambda) \in \langle \text{supp}_{Q_{\mathfrak{h}} \times \mathbb{Z}^n}(\mathfrak{L}) \rangle$. By Corollary 3.3.6, $\Delta := \text{supp}_{Q_{\mathfrak{h}}}(\mathfrak{L})$ is an irreducible finite root system. Let Φ be a base of Δ , and for each $\alpha \in \Phi$, we take $\lambda_\alpha \in \mathbb{Z}^n$ such that $\mathfrak{L}_\alpha^{\lambda_\alpha} \neq \{0\}$. Since Φ is a \mathbb{Z}^n -basis of $Q_{\mathfrak{h}}$, we can take $t = (t_1, \dots, t_n) \in \text{Hom}(Q_{\mathfrak{h}}, \mathbb{Z}^n)$ satisfying $t(\alpha) = \lambda_\alpha$ for $\alpha \in \Phi$. Then since $(\mathfrak{L}^{(t)})_\alpha^0 = \mathfrak{L}_\alpha^{t(\alpha)} \neq \{0\}$ for $\alpha \in \Phi$, $(Q_{\mathfrak{h}}, 0) \subseteq \langle \text{supp}_{Q_{\mathfrak{h}} \times \mathbb{Z}^n}(\mathfrak{L}^{(t)}) \rangle$. Thus, we have

$$\langle \text{supp}_{Q_{\mathfrak{h}} \times \mathbb{Z}^n}(\mathfrak{L}^{(t)}) \rangle = \{(\alpha, \lambda) \mid \alpha \in Q_{\mathfrak{h}}, \lambda \in \langle \text{supp}_{\mathbb{Z}^n}(\mathfrak{L}^{(t)}) \rangle\},$$

and then we have

$$\langle \text{supp}_{Q_{\mathfrak{h}} \times \mathbb{Z}^n}(\mathfrak{L}) \rangle = \{(\alpha, \lambda + t(\alpha)) \mid \alpha \in Q_{\mathfrak{h}}, \lambda \in \langle \text{supp}_{\mathbb{Z}^n}(\mathfrak{L}^{(t)}) \rangle\}. \quad (30)$$

Next, we define $u = (u_1, \dots, u_n) \in \text{Hom}(Q_{\mathfrak{h}'}, \mathbb{Z}^n)$ as

$$u(\hat{\psi}(\alpha)) = p(\alpha, t(\alpha))$$

for $\alpha \in Q_{\mathfrak{h}}$. Since $\mathfrak{L} = \mathfrak{L}^{(t)}$ and $\mathfrak{L}' = \mathfrak{L}'^{(u)}$ as Lie algebras, we can consider ψ as a Lie algebra isomorphism from $\mathfrak{L}^{(t)}$ onto $\mathfrak{L}'^{(u)}$. Let $\alpha \in Q_{\mathfrak{h}}$ and $\lambda \in \langle \text{supp}_{\mathbb{Z}^n}(\mathfrak{L}^{(t)}) \rangle$. Then $(\alpha, \lambda + t(\alpha)) \in \langle \text{supp}_{Q_{\mathfrak{h}} \times \mathbb{Z}^n}(\mathfrak{L}) \rangle$ by (30). Thus, we have

$$\psi((\mathfrak{L}^{(t)})_\alpha^\lambda) = \psi(\mathfrak{L}_\alpha^{\lambda+t(\alpha)}) = \mathfrak{L}'_{\hat{\psi}(\alpha)}^{p(\alpha, \lambda+t(\alpha))} = (\mathfrak{L}'^{(u)})_{\hat{\psi}(\alpha)}^{p((0, \lambda))},$$

and this means

$$\psi((\mathfrak{L}^{(t)})^\lambda) = (\mathfrak{L}'^{(u)})^{p((0, \lambda))},$$

for $\lambda \in \langle \text{supp}_{\mathbb{Z}^n}(\mathfrak{L}^{(t)}) \rangle$. Since the map $\lambda \mapsto p((0, \lambda))$ is additive, the Lie algebra isomorphism ψ is indeed a \mathbb{Z}^n -support-isomorphism from $\mathfrak{L}^{(t)}$ to $\mathfrak{L}'^{(u)}$. We define $\tilde{\tau}_i \in \text{Aut}(\mathfrak{g})$, $\tilde{\tau}'_i \in \text{Aut}(\mathfrak{g}')$ for $1 \leq i \leq n$ as $\tilde{\tau}_i(x_\alpha) = \zeta_{m_i}^{-t_i(\alpha)} x_\alpha$ for $\alpha \in Q_{\mathfrak{h}}, x_\alpha \in \mathfrak{g}_\alpha$, and $\tilde{\tau}'_i(y_\beta) = \zeta_{m'_i}^{-u_i(\beta)} y_\beta$ for $\alpha \in Q_{\mathfrak{h}'}, y_\beta \in \mathfrak{g}'_\beta$. By Lemma 4.2.4, $\mathfrak{L}^{(t)} \cong_{Q_{\mathfrak{h}} \times \mathbb{Z}^n} L_{\mathfrak{m}}(\mathfrak{g}, \tilde{\tau}\sigma, \mathfrak{h})$ and $\mathfrak{L}'^{(u)} \cong_{Q_{\mathfrak{h}'} \times \mathbb{Z}^n} L_{\mathfrak{m}'}(\mathfrak{g}', \tilde{\tau}'\sigma', \mathfrak{h}')$ where $\tilde{\tau} = (\tilde{\tau}_1, \dots, \tilde{\tau}_n)$ and $\tilde{\tau}' = (\tilde{\tau}'_1, \dots, \tilde{\tau}'_n)$. Therefore, we have

$$L_{\mathfrak{m}}(\mathfrak{g}, \tilde{\tau}\sigma, \mathfrak{h}) \cong_{\mathbb{Z}^n - \text{su}} L_{\mathfrak{m}'}(\mathfrak{g}', \tilde{\tau}'\sigma', \mathfrak{h}'),$$

and then from Lemma 4.1.2, there exist $P \in \text{GL}_n(\mathbb{Z})$ and a Lie algebra isomorphism $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}'$ such that

$$\tilde{\tau}'\sigma' = \varphi(\tilde{\tau}\sigma)^P \varphi^{-1}. \quad (31)$$

Using a similar argument as the proof of Proposition 4.1.3, we can suppose that $\varphi(\mathfrak{h}) = \mathfrak{h}'$. Under this assumption we define $\hat{\varphi} : Q_{\mathfrak{h}} \rightarrow Q_{\mathfrak{h}'}$ as $\langle \hat{\varphi}(\alpha), \varphi(h) \rangle = \langle \alpha, h \rangle$. Adding this, we set $P^{-1} = (q_{ij})$, and finally we define $s = (s_1, \dots, s_n) \in \text{Hom}(Q_{\mathfrak{h}}, \mathbb{Q}^n)$ as

$$s_i = \frac{1}{m_i} t_i - \sum_j \frac{q_{ij}}{m'_j} u_j \circ \hat{\varphi}.$$

If τ_i is defined as (29),

$$\tau_i(x_\alpha) = \left(\prod_j \zeta_{m'_j}^{q_{ij} u_j(\hat{\varphi}(\alpha))} \right) \cdot \zeta_{m_i}^{-t_i(\alpha)} x_\alpha = \varphi^{-1} \circ \left(\prod_j \tilde{\tau}'_j{}^{-q_{ij}} \right) \circ \varphi \circ \tilde{\tau}_i(x_\alpha)$$

for $\alpha \in Q_{\mathfrak{h}}, x_\alpha \in \mathfrak{g}_\alpha$. Thus, $\tau = \left(\varphi^{-1} \tilde{\tau}'^{-P^{-1}} \varphi \right) \tilde{\tau}$. Then we have from (31) that

$$\begin{aligned} \sigma' &= \tilde{\tau}'^{-1}(\varphi(\tilde{\tau}\sigma)^P \varphi^{-1}) = \varphi(\varphi^{-1} \tilde{\tau}'^{-1} \varphi)(\tilde{\tau}\sigma)^P \varphi^{-1} \\ &= \varphi \left(\left(\varphi^{-1} \tilde{\tau}'^{-P^{-1}} \varphi \right) \tilde{\tau}\sigma \right)^P \varphi^{-1} = \varphi(\tau\sigma)^P \varphi^{-1}, \end{aligned}$$

and (b) follows.

“(b) \Rightarrow (c)” Suppose that $s \in \text{Hom}(Q_{\mathfrak{h}}, \mathbb{Q}^n)$, $P \in \text{GL}_n(\mathbb{Z})$ and φ satisfy (b). For $1 \leq i \leq n$, let $a_i \in \mathbb{Z}_{>0}$ be a positive number satisfying

$$a_i s_i(\alpha) \in \mathbb{Z} \quad \text{for all } \alpha \in Q_{\mathfrak{h}},$$

and let $\tilde{\mathbf{m}} = (a_1 m_1, \dots, a_n m_n) \in \mathbb{Z}_{>0}^n$. From Lemma 4.2.3, there exists a monomorphism $\rho_1 : \langle \text{supp}_{\mathbb{Z}^n}(\mathfrak{L}) \rangle \rightarrow \mathbb{Z}^n$ such that

$$\mathfrak{L}_{(\rho_1)} \cong_{Q_{\mathfrak{h}} \times \mathbb{Z}^n} L_{\tilde{\mathbf{m}}}(\mathfrak{g}, \sigma, \mathfrak{h}). \quad (32)$$

If we set $t = (a_1 m_1 s_1, \dots, a_n m_n s_n)$, we have using Lemma 4.2.4 that

$$L_{\tilde{\mathbf{m}}}(\mathfrak{g}, \sigma, \mathfrak{h})^{(t)} \cong_{Q_{\mathfrak{h}} \times \mathbb{Z}^n} L_{\tilde{\mathbf{m}}}(\mathfrak{g}, \tau\sigma, \mathfrak{h}). \quad (33)$$

Using Lemma 4.2.3 again, there exists $\rho_2 : \langle \text{supp}_{\mathbb{Z}^n}(L_{\tilde{\mathbf{m}}}(\mathfrak{g}, \tau\sigma, \mathfrak{h})) \rangle \rightarrow \mathbb{Z}^n$ such that

$$L_{\tilde{\mathbf{m}}}(\mathfrak{g}, \tau\sigma, \mathfrak{h})_{(\rho_2)} \cong_{Q_{\mathfrak{h}} \times \mathbb{Z}^n} L_{\mathbf{m}'}(\mathfrak{g}, (\tau\sigma)^P, \mathfrak{h}). \quad (34)$$

By the assumptions and the definition of a multiloop Lie algebra, it is easily seen that

$$\mathfrak{L}' \cong_{\mathbb{Z}\text{-ig}} L_{\mathbf{m}'}(\mathfrak{g}, (\tau\sigma)^P, \mathfrak{h}). \quad (35)$$

Then from (32), (33), (34) and (35), $\mathfrak{L}_0 = \mathfrak{L}$, $\mathfrak{L}_1 = \mathfrak{L}_{(\rho_1)}$, $\mathfrak{L}_2 = \mathfrak{L}_1^{(t)}$, $\mathfrak{L}_3 = \mathfrak{L}_{2(\rho_2)}$ is the finite sequence satisfying (c).

“(c) \Rightarrow (a)” Suppose that the sequence $\mathfrak{L}_0, \mathfrak{L}_1, \dots, \mathfrak{L}_p$ satisfies (c). Then $\mathfrak{L}_p \cong_{\text{ig}} \mathfrak{L}'$ by Proposition 4.1.3, and we have $\mathfrak{L}_p \cong_{\text{supp}} \mathfrak{L}'$ from Lemma 2.2.2 (a). Thus, it suffices to show that $\mathfrak{L} \cong_{\text{supp}} \mathfrak{L}_{(\rho)}$ for a monomorphism $\rho : \langle \text{supp}_{\mathbb{Z}^n}(\mathfrak{L}) \rangle \rightarrow \mathbb{Z}^n$, and $\mathfrak{L} \cong_{\text{supp}} \mathfrak{L}^{(s)}$ for $s \in \text{Hom}(Q_{\mathfrak{h}}, \mathbb{Z}^n)$. The first statement is proved as follows. Since $\mathfrak{L} = \mathfrak{L}_{(\rho)}$ as a Lie algebra, an identity on \mathfrak{L} induces a Lie algebra isomorphism from \mathfrak{L} onto $\mathfrak{L}_{(\rho)}$. Then since this isomorphism sends $\mathfrak{L}_\alpha^\lambda$ to $(\mathfrak{L}_{(\rho)})_\alpha^{\rho(\lambda)}$ for $(\alpha, \lambda) \in \langle \text{supp}_{Q_{\mathfrak{h}} \times \mathbb{Z}^n}(\mathfrak{L}) \rangle$, this isomorphism is indeed a support-isomorphism. To show the second statement, we consider a Lie algebra isomorphism $\mathfrak{L} \rightarrow \mathfrak{L}^{(s)}$ induced by an identity on \mathfrak{L} . This isomorphism sends $\mathfrak{L}_\alpha^\lambda$ to $(\mathfrak{L}^{(s)})_\alpha^{\lambda - s(\alpha)}$ for $\alpha \in Q_{\mathfrak{h}}, \lambda \in \mathbb{Z}^n$. Since the map

$$Q_{\mathfrak{h}} \times \mathbb{Z}^n \ni (\alpha, \lambda) \mapsto (\alpha, \lambda - s(\alpha)) \in Q_{\mathfrak{h}} \times \mathbb{Z}^n$$

is a group isomorphism, this isomorphism is indeed an isograded-isomorphism. Then $\mathfrak{L} \cong_{\text{supp}} \mathfrak{L}^{(s)}$ by Lemma 2.2.2 (a). \square

5. THE RELATION AMONG MULTILOOP LIE ALGEBRAS, LIE TORI AND EXTENDED AFFINE LIE ALGEBRAS

E. Neher has introduced in [N] to construct EALAs from Lie tori. In this chapter, we consider the construction of EALAs from multiloop Lie algebras that are not necessarily Lie tori.

5.1. Lie \mathbb{Z}^n -tori. In this subsection, using the results of [ABFP2] we give a necessary and sufficient condition for a multiloop Lie algebra to be support-isomorphic to some Lie \mathbb{Z}^n -torus, which is defined to be a $Q \times \mathbb{Z}^n$ -graded Lie algebra for some root lattice Q satisfying several axioms. Since it is not needed for the purpose of this article, we do not state the definition of a Lie \mathbb{Z}^n -torus. (For the definition, see [ABFP2, Definition 1.1.6]).

If Δ is an irreducible finite root system, we define an *indivisible* root system Δ_{ind} and an *enlarged* root system Δ_{en} as

$$\Delta_{\text{ind}} = \{\alpha \in \Delta \mid \frac{1}{2}\alpha \notin \Delta\},$$

and

$$\Delta_{\text{en}} = \begin{cases} \Delta \cup \{2\alpha \mid \alpha : \text{short root of } \Delta\} & \text{if } \Delta \text{ has type } B_l, l \geq 1; \\ \Delta & \text{otherwise.} \end{cases}$$

By [ABFP2, Proposition 3.2.5], we have the following proposition:

Proposition 5.1.1. *Let $\mathfrak{L} = L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ be a multiloop Lie algebra. Then \mathfrak{L} is a Lie \mathbb{Z}^n -torus if and only if the following condition (A0) \sim (A3) are satisfied:*

- (A0) $\mathbf{m} = \text{ord}(\sigma)$.
- (A1) \mathfrak{g}^σ is a simple Lie algebra.
- (A2) If $\bar{0} \neq \bar{\lambda} \in \text{supp}_{\bar{\Lambda}_{\mathbf{m}}}(\mathfrak{g})$, then $\mathfrak{g}^{\bar{\lambda}} \cong U^{\bar{\lambda}} \oplus V^{\bar{\lambda}}$ as a \mathfrak{g}^σ -module, where \mathfrak{g}^σ acts trivially on $U^{\bar{\lambda}}$ and either $V^{\bar{\lambda}} = \{0\}$ or $V^{\bar{\lambda}}$ is irreducible of dimension > 1 and the weights of $V^{\bar{\lambda}}$ relative to \mathfrak{h} are contained in $(\Delta)_{\text{en}} \cup \{0\}$ where Δ is a root system of \mathfrak{g} relative to \mathfrak{h} .
- (A3) $|\langle \{\sigma_1, \dots, \sigma_n\} \rangle| = \prod_{1 \leq i \leq n} \text{ord}(\sigma_i)$.

If \mathfrak{L} satisfies the condition in the above proposition, we call \mathfrak{L} a *multiloop Lie \mathbb{Z}^n -torus* determined by $\mathfrak{g}, \sigma, \mathfrak{h}$. \square

Later, we use the following simple lemmas about a finite dimensional simple Lie algebra.

Lemma 5.1.2. *Let \mathfrak{g} be a finite dimensional simple Lie algebra, $\sigma \in \text{Aut}_{\text{cfo}}^n(\mathfrak{g})$ and $\mathbf{m} \in \mathbb{Z}_{>0}^n$ such that $\sigma^{\mathbf{m}} = \text{id}$ and $\mathfrak{g}^\sigma \neq \{0\}$, and let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g}^σ . We define the $Q_{\mathfrak{h}} \times \bar{\Lambda}_{\mathbf{m}}$ -grading on \mathfrak{g} as in (2). Suppose that $\alpha \in \Delta := \text{supp}_{Q_{\mathfrak{h}}}(\mathfrak{g}) \setminus \{0\}$ and $\bar{\lambda} \in \bar{\Lambda}_{\mathbf{m}}$ satisfy $\mathfrak{g}_{\alpha}^{\bar{\lambda}} \neq \{0\}$. Then*

- (a) $\mathfrak{g}_{2\alpha}^{2\bar{\lambda}} = \{0\}$,
- (b) $\dim \mathfrak{g}_{\alpha}^{\bar{\lambda}} = 1$.

Proof. Let $\{x_{\alpha}^{\bar{\lambda}}, x_{-\alpha}^{-\bar{\lambda}}, h_{\alpha}\}$ be a $\mathfrak{sl}_2(k)$ -triple with respect to $(\alpha, \bar{\lambda})$, and we denote by $S_{\alpha}^{\bar{\lambda}}$ the subalgebra spanned by these elements. Recall that $[\mathfrak{g}_{\alpha}^{\bar{\lambda}}, \mathfrak{g}_{-\alpha}^{-\bar{\lambda}}] \subseteq kh_{\alpha}$ (cf. subsection 3.3).

(a) Suppose that $\mathfrak{g}_{2\alpha}^{2\bar{\lambda}} \neq \{0\}$ and take $0 \neq z \in \mathfrak{g}_{2\alpha}^{2\bar{\lambda}}$. Note that Δ is a irreducible finite

root system by Proposition 3.3.5. Since $\text{ad}(x_\alpha^\lambda)(z) = \{0\}$ and z is an eigenvector for $\text{ad}(h_\alpha)$ with eigenvalue 4,

$$V := \sum_{0 \leq i} (\text{ad}(S_\alpha^\lambda))^i (x_{2\alpha}^{2\lambda})$$

is a 5-dimensional irreducible S_α^λ -module. On the other hand, since

$$(\text{ad}(x_{-\alpha}^{-\lambda}))^2(z) \subseteq [x_{-\alpha}^{-\lambda}, \mathfrak{g}_\alpha^\lambda] \subseteq kh_\alpha,$$

V contains h_α . Then we have $S_\alpha^\lambda \subseteq V$, and this contradicts the irreducibility of V .

(b) For $w \in \mathfrak{g}_\alpha^\lambda$, $[x_\alpha^\lambda, w] = 0$ by (a). If $[x_{-\alpha}^{-\lambda}, w] = 0$, then we have

$$[h_\alpha, w] = [[x_\alpha^\lambda, x_{-\alpha}^{-\lambda}], w] = 0,$$

and $w = 0$ follows. Therefore, $\text{ad}(x_{-\alpha}^{-\lambda})$ is an injective k -linear map from $\mathfrak{g}_\alpha^\lambda$ to 1-dimensional space kh_α , thus (b) follows. \square

The following lemma follows from [ABFP2, Lemma 3.2.4]:

Lemma 5.1.3. *Let \mathfrak{g} be a finite dimensional simple Lie algebra and W is a finite dimensional \mathfrak{g} -module. We set Δ be a root system of \mathfrak{g} relative to a Cartan subalgebra \mathfrak{h} . If the weights of W relative to \mathfrak{h} are contained in $\Delta_{\text{en}} \cup \{0\}$ and $\dim W_\alpha \leq 1$ for $\alpha \in \Delta_{\text{en}}$, then $W = U \oplus V$ where \mathfrak{g} acts trivially on U and either $V = \{0\}$ or V is irreducible of dimension > 1 . \square*

Theorem 5.1.4. *Let $\mathfrak{L} = L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ be a multiloop Lie algebra of nullity n . Then \mathfrak{L} is support-isomorphic to some multiloop Lie \mathbb{Z}^n -torus if and only if $\mathfrak{g}^\sigma \neq \{0\}$.*

Proof. First, we show the “only if” part. Suppose that $\mathfrak{L} \cong_{\text{supp}} \mathcal{L}$ for a multiloop Lie \mathbb{Z}^n -torus \mathcal{L} . Then $\mathfrak{h} = \mathfrak{L}_0^0 \cong \mathcal{L}_0^0 \neq \{0\}$. Thus, $\mathfrak{g}^\sigma \neq \{0\}$ follows. Next, we show the “if” part. Suppose that $\mathfrak{g}^\sigma \neq \{0\}$. Let $\Delta = \text{supp}_{Q_{\mathfrak{h}}}(\mathfrak{L}) \setminus \{0\}$ and $Q_{\mathfrak{h}} = \sum_{\alpha \in \Delta} \mathbb{Z}\alpha$. By Corollary 3.3.6, Δ is an irreducible finite root system. Take an arbitrary base Φ of Δ and choose $\lambda_\alpha \in \mathbb{Z}^n$ for each $\alpha \in \Phi$ such that $\mathfrak{g}_\alpha^{\overline{\lambda_\alpha(\sigma, m)}} \neq \{0\}$. Since Φ is a \mathbb{Z} -basis of $Q_{\mathfrak{h}}$, we can take $s = (s_1, \dots, s_n) \in \text{Hom}(Q_{\mathfrak{h}}, \mathbb{Z}^n)$ such that

$$s(\alpha) = \lambda_\alpha \quad \text{for } \alpha \in \Phi.$$

We define $\tau_i \in \text{Aut}(\mathfrak{g})$ for $1 \leq i \leq n$ as

$$\tau_i(x_\alpha) = \zeta_{m_i}^{-s_i(\alpha)} x_\alpha$$

for $\alpha \in Q_{\mathfrak{h}}, x_\alpha \in \mathfrak{g}_\alpha$. Let $\tau = (\tau_1, \dots, \tau_n)$ and $\tilde{\sigma} = \tau\sigma$. Then $\mathfrak{L}^{(s)}$ is $Q_{\mathfrak{h}} \times \mathbb{Z}^n$ -graded-isomorphic to $L_{\mathbf{m}}(\mathfrak{g}, \tilde{\sigma}, \mathfrak{h})$ by Lemma 4.2.4. Then

$$\mathfrak{g}_\alpha^{\tilde{\sigma}} \cong (\mathfrak{L}^{(s)})_\alpha^0 = \mathfrak{L}_\alpha^{s(\alpha)} \cong \mathfrak{g}_\alpha^{\overline{s(\alpha)(\sigma, m)}}$$

for $\alpha \in \Delta$. Thus, we have that

$$\pm\Phi \subseteq \text{supp}_{Q_{\mathfrak{h}}}(\mathfrak{g}^{\tilde{\sigma}}) \subseteq \Delta$$

by the construction of s . Since $\mathfrak{g}^{\tilde{\sigma}}$ is reductive, $\text{supp}_{Q_{\mathfrak{h}}}(\mathfrak{g}^{\tilde{\sigma}}) = \Delta_{\text{ind}}$, that is, $\mathfrak{g}^{\tilde{\sigma}}$ is a simple Lie algebra with the root system Δ_{ind} . Using [ABFP2, Proposition 5.1.3], we can take $P \in \text{GL}_n(\mathbb{Z})$ such that

$$|\langle \{\tilde{\sigma}_1, \dots, \tilde{\sigma}_n\} \rangle| = \prod_{1 \leq i \leq n} \text{ord}(\tilde{\sigma}_i^P)$$

where we set $\tilde{\sigma}^P = (\tilde{\sigma}_1^P, \dots, \tilde{\sigma}_n^P)$. We prove that $L_{\text{ord}}(\tilde{\sigma}^P) := L_{\text{ord}}(\tilde{\sigma}^P)(\mathfrak{g}, \tilde{\sigma}^P, \mathfrak{h})$ is a multiloop Lie \mathbb{Z}^n -torus. It suffices to show that $L_{\text{ord}}(\tilde{\sigma}^P)$ satisfies (A0) \sim (A3) in Proposition 5.1.1. (A0) and (A3) are trivial and (A1) has been already shown. Since the weights of \mathfrak{g} relative to \mathfrak{h} are contained in $\Delta \cup \{0\} \subseteq (\Delta_{\text{ind}})_{\text{en}} \cup \{0\}$ and $\dim \mathfrak{g}_{\alpha}^{\bar{\lambda}(\tilde{\sigma}, m)} \leq 1$ for $\alpha \in \Delta, \bar{\lambda} \in \bar{\Lambda}_m$ by Lemma 5.1.2 (b), (A2) follows from Lemma 5.1.3. Thus, $L_{\text{ord}}(\tilde{\sigma}^P)$ is a multiloop Lie \mathbb{Z}^n -torus. Finally, $\mathfrak{L} \cong_{\text{supp}} L_{\text{ord}}(\tilde{\sigma}^P)$ follows from Theorem 4.2.5. \square

5.2. Extended affine Lie algebras. In this subsection, we consider the construction of an extended affine Lie algebra (EALA, for short) from a multiloop Lie algebra.

First, we recall the definition of an EALA. (The following version of the definition is introduced in [N]).

Definition 5.2.1. An extended affine Lie algebra over k of nullity n is a triple $(E, H, (\mid))$, where E is a Lie algebra over k , H is a subalgebra of E , and (\mid) is a bilinear form on E , satisfying the following conditions (EA1) \sim (EA6):

- (EA1) (\mid) is a non-degenerate invariant symmetric bilinear form.
- (EA2) H is a nontrivial finite-dimensional self-centralizing and ad-diagonalizable subalgebra of E .

Let $R = \text{supp}_{H^*}(E)$ where we consider a root space decomposition of E with respect to H . We define (\mid) on H^* in a similar way as (16) and let $R^0 = \{\alpha \in R \mid (\alpha|\alpha) = 0\}$.

- (EA3) For $\alpha \in R \setminus R^0$ and $x_\alpha \in E_\alpha$, $\text{ad}(x_\alpha)$ is locally nilpotent.
- (EA4) $R \setminus R^0$ is irreducible.
- (EA5) If E_c is a subalgebra in E generated by $\{E_\alpha \mid \alpha \in R \setminus R^0\}$, then $\{e \in E \mid [e, E_c] = 0\} \subseteq E_c$.
- (EA6) $\langle R^0 \rangle \subseteq H^*$ is a free abelian group of rank n .

If $(E, H, (\mid))$ is an EALA, we also say that E is an EALA for short.

The following definition is introduced in [AF]:

Definition 5.2.2. Suppose that $(E, H, (\mid))$ and $(E', H', (\mid)')$ are EALAs. We say $(E, H, (\mid))$ and $(E', H', (\mid)')$ are isomorphic if there exists a Lie algebra isomorphism $\chi : E \rightarrow E'$ such that

$$\chi(H) = H' \text{ and } (\chi(x)|\chi(y))' = a(x|y) \text{ for some } a \in k.$$

Alternatively, in that case we say E and E' are isomorphic as EALAs.

In [N], E. Neher introduced a construction of a family of EALAs from a Lie Λ -torus where Λ is a free abelian group of finite rank, and he proved that any EALA is constructed by this. Observing that construction, we can see that it can be applied to some Lie algebras that are not Lie tori. Indeed, we show in Proposition 5.2.4 that if a Lie algebra \mathfrak{L} with subalgebra \mathfrak{h} and a bilinear form (\mid) satisfies the following conditions (L1) \sim (L4), we can construct an EALA from them by the same construction with Neher's construction:

- (L1) $\mathfrak{L} = \bigoplus_{\lambda \in \Lambda} \mathfrak{L}^\lambda$ is a graded-central-simple Λ -graded Lie algebra where Λ is a free abelian group of finite rank n .
- (L2) A rank of the central grading group $\Gamma = \Gamma_\Lambda(\mathfrak{L})$ is n (cf. Definition 2.1.2).

- (L3) (\mid) is a non-degenerate invariant symmetric Λ -graded bilinear form. (\mid) is Λ -graded means that $(x|y) = 0$ for $x \in \mathfrak{L}^\lambda, y \in \mathfrak{L}^\mu$ if $\lambda + \mu \neq 0$.
- (L4) $\mathfrak{h} \subseteq \mathfrak{L}^0$, \mathfrak{h} is abelian, ad-diagonalizable on \mathfrak{L} and $\Delta := \text{supp}_{\mathfrak{h}^*}(\mathfrak{L}) \setminus \{0\}$ is an irreducible finite root system where we consider a root space decomposition of \mathfrak{L} with respect to \mathfrak{h} .

First, we roughly describe this construction for the later explanation. (For more precise description, see [N] or [AF]).

Construction 5.2.3. Let $\mathfrak{L}, \mathfrak{h}, (\mid)$ satisfy (L1) \sim (L4). For $\theta \in \text{Hom}(\Lambda, k)$, we define a degree derivation ∂_θ of \mathfrak{L} by

$$\partial_\theta(x^\lambda) = \theta(\lambda)x^\lambda$$

for $\lambda \in \Lambda, x^\lambda \in \mathfrak{L}^\lambda$. We put

$$\text{SCDer}(\mathfrak{L}) = \bigoplus_{\mu \in \Gamma} C(\mathfrak{L})^\mu \cdot \{\partial_\theta \mid \theta(\mu) = 0\}$$

where $C(\mathfrak{L})$ is a centroid of \mathfrak{L} . Then $\text{SCDer}(\mathfrak{L})$ is a subalgebra of $\text{Der}(\mathfrak{L})$. To construct an EALA from \mathfrak{L} , we need following two ingredients:

- (i) Let $\mathfrak{D} = \bigoplus_{\mu \in \Gamma} \mathfrak{D}^\mu$ be a Γ -graded subalgebra of $\text{SCDer}(\mathfrak{L})$ such that an evaluation map $\text{ev} : \Lambda \rightarrow (\mathfrak{D}^0)^*$ defined by $\text{ev}(\lambda)(\partial_\theta) = \theta(\lambda)$ is injective. Let $\mathfrak{C} = \bigoplus_{\mu \in \Gamma} (\mathfrak{D}^\mu)^*$ and consider \mathfrak{C} as a \mathfrak{D} -module by a contragredient action. We give \mathfrak{C} a Γ -grading by $\mathfrak{C}^\mu = (\mathfrak{D}^{-\mu})^*$.
- (ii) Let $\tau : \mathfrak{D} \times \mathfrak{D} \rightarrow \mathfrak{C}$ be a graded invariant 2-cocycle, i.e.

$$\tau(\mathfrak{D}^{\mu_1}, \mathfrak{D}^{\mu_2}) \subseteq \mathfrak{C}^{\mu_1 + \mu_2}, \quad \tau(d_1, d_2)(d_3) = \tau(d_2, d_3)(d_1) \quad \text{for } d_i \in \mathfrak{D}$$

$$\text{such that } \tau(\mathfrak{D}, \mathfrak{D}^0) = 0.$$

Then $E(\mathfrak{L}, \mathfrak{D}, \tau) := \mathfrak{L} \oplus \mathfrak{C} \oplus \mathfrak{D}$ is a Lie algebra with respect to the product

$$\begin{aligned} [x_1 + c_1 + d_1, x_2 + c_2 + d_2] = & ([x_1, x_2] + d_1(x_2) - d_2(x_1)) \\ & + (\sigma_{\mathfrak{D}}(x_1, x_2) + d_1 \cdot c_2 - d_2 \cdot c_1 + \tau(d_1, d_2)) \\ & + [d_1, d_2] \end{aligned}$$

for $x_i \in \mathfrak{L}, c_i \in \mathfrak{C}, d_i \in \mathfrak{D}$ where $\sigma_{\mathfrak{D}} : \mathfrak{L} \times \mathfrak{L} \rightarrow \mathfrak{C}$ is defined by $\sigma_{\mathfrak{D}}(x, y)(d) = (d(x)|y)$ for $x, y \in \mathfrak{L}, d \in \mathfrak{D}$. We can define a bilinear form (\mid) on $E(\mathfrak{L}, \mathfrak{D}, \tau)$ by

$$(x_1 + c_1 + d_1 | x_2 + c_2 + d_2) = (x_1 | x_2) + d_1(c_2) + d_2(c_1).$$

Then we have the following proposition:

Proposition 5.2.4. *If $\mathfrak{L}, \mathfrak{h}, (\mid)$ satisfy the condition (L1) \sim (L4), then $E(\mathfrak{L}, \mathfrak{D}, \tau)$ constructed in Construction 5.2.3 is an EALA of nullity n with respect to the form (\mid) and the subalgebra $H = \mathfrak{h} \oplus \mathfrak{C}^0 \oplus \mathfrak{D}^0$.*

Proof. We only prove (EA5) in Definition 5.2.1. We write $E = E(\mathfrak{L}, \mathfrak{D}, \tau), E_c = E(\mathfrak{L}, \mathfrak{D}, \tau)_c$. Since $\{e \in E \mid [e, E_c] = 0\} = \mathfrak{C}$, we show that $\mathfrak{C} \subseteq E_c$. Since $E_c \subseteq [\mathfrak{L}, \mathfrak{L}]$, it suffices to show that $\mathfrak{C} \subseteq [\mathfrak{L}, \mathfrak{L}]$. By (L1) and [ABFP1, Lemma 4.3.5 and 4.3.8], $C(\mathfrak{L}) \cong k[\Gamma]$ as a Γ -graded algebra where $k[\Gamma]$ is a group algebra of Γ over k . Using the above isomorphism, we write $C(\mathfrak{L}) = \bigoplus_{\mu \in \Gamma} k t^\mu$ where $t^{\mu_1} \cdot t^{\mu_2} = t^{\mu_1 + \mu_2}$ for $\mu_1, \mu_2 \in \Gamma$. Note that

$$(t^\mu \cdot x | y) = (x | t^\mu \cdot y) \tag{36}$$

for $x, y \in \mathfrak{L}, \mu \in \Gamma$ since \mathfrak{L} is perfect. We set a Γ -graded k -linear space $\text{CDer}(\mathfrak{L})$ as

$$\text{CDer}(\mathfrak{L}) = \bigoplus_{\mu \in \Gamma} t^\mu \cdot \{\partial_\theta \mid \theta \in \text{Hom}(\Lambda, k)\}$$

and set $\text{CDer}(\mathfrak{L})^{\text{gr}*} = \bigoplus_{\mu \in \Gamma} (\text{CDer}(\mathfrak{L})^\mu)^*$. We give $\text{CDer}(\mathfrak{L})^{\text{gr}*}$ a Γ -grading by $(\text{CDer}(\mathfrak{L})^{\text{gr}*})^\mu = (\text{CDer}(\mathfrak{L})^{-\mu})^*$. By (L2), we can take a \mathbb{Z} -basis $\{\nu_1, \dots, \nu_n\}$ of Γ . If we define $c_i^\mu \in (\text{CDer}(\mathfrak{L})^{\text{gr}*})^\mu$ for $1 \leq i \leq n, \mu \in \Gamma$ as

$$c_i^{\mu_1}(t^{\mu_2} \cdot \partial_\theta) = \begin{cases} \theta(\nu_i) & \text{if } \mu_1 + \mu_2 = 0 \\ 0 & \text{otherwise,} \end{cases}$$

then it is easily seen that $\{c_i^\mu \mid 1 \leq i \leq n, \mu \in \Gamma\}$ is a k -basis of $\text{CDer}(\mathfrak{L})^{\text{gr}*}$. We define a k -linear map $\sigma : \mathfrak{L} \otimes \mathfrak{L} \rightarrow \text{CDer}(\mathfrak{L})^{\text{gr}*}$ by

$$\sigma(x \otimes y)(d) = (d(x)|y)$$

for $x, y \in \mathfrak{L}, d \in \text{CDer}(\mathfrak{L})$. We choose $x, y \in \mathfrak{L}$ such that $(x|y) = 1$. Then it is easily checked using (36) that

$$\sigma((t^{\nu_i} \cdot x) \otimes (t^{-\nu_i - \mu} \cdot y)) - \sigma(x \otimes (t^{-\mu} \cdot y)) = c_i^\mu$$

for $1 \leq i \leq n, \mu \in \Gamma$. Let $\eta : \text{CDer}(\mathfrak{L})^{\text{gr}*} \rightarrow \mathfrak{C}$ be a restriction on \mathfrak{D} , and then we have

$$[t^{\nu_i} \cdot x, t^{-\nu_i - \mu} \cdot y] - [x, t^{-\mu} \cdot y] = \eta(c_i^\mu).$$

for $1 \leq i \leq n, \mu \in \Gamma$. Since \mathfrak{C} is spanned by $\{\eta(c_i^\mu) \mid 1 \leq i \leq n, \mu \in \Gamma\}$, $\mathfrak{C} \subseteq [\mathfrak{L}, \mathfrak{L}]$ follows. \square

For $(\mathfrak{L}, \mathfrak{h}, (\mid))$ satisfying the conditions (L1) \sim (L4), we set

$$\mathcal{P}(\mathfrak{L}) = \{(\mathfrak{D}, \tau) \mid \mathfrak{D}, \tau \text{ are as in (i), (ii) in Construction 5.2.3}\}.$$

Note that $\mathcal{P}(\mathfrak{L})$ does not depend on \mathfrak{h} or (\mid) .

We use the following notation: suppose that $(\mathfrak{L}, \mathfrak{h}, (\mid))$ and $(\mathfrak{L}', \mathfrak{h}', (\mid)')$ satisfy the conditions (L1) \sim (L4). Then we will write

$$(\mathfrak{L}, \mathfrak{h}, (\mid)) \sim_{\text{EALA}} (\mathfrak{L}', \mathfrak{h}', (\mid)') \text{ (or } \mathfrak{L} \sim_{\text{EALA}} \mathfrak{L}' \text{ for short)}$$

if there exists a bijection $\mathcal{P}(\mathfrak{L}) \rightarrow \mathcal{P}(\mathfrak{L}')$ such that $E(\mathfrak{L}, \mathfrak{D}, \tau)$ is isomorphic as EALAs to $E(\mathfrak{L}', \mathfrak{D}', \tau')$ where $(\mathfrak{D}', \tau') \in \mathcal{P}(\mathfrak{L}')$ is the image of $(\mathfrak{D}, \tau) \in \mathcal{P}(\mathfrak{L})$ under the bijection. In other words, $\mathfrak{L} \sim_{\text{EALA}} \mathfrak{L}'$ means that $\{E(\mathfrak{L}, \mathfrak{D}, \tau) \mid (\mathfrak{D}, \tau) \in \mathcal{P}(\mathfrak{L})\}$ and $\{E(\mathfrak{L}', \mathfrak{D}', \tau') \mid (\mathfrak{D}', \tau') \in \mathcal{P}(\mathfrak{L}')\}$ are coincides up to isomorphisms as EALAs.

Using the above notation, we have the following:

Lemma 5.2.5. *Suppose that $(\mathfrak{L}, \mathfrak{h}, (\mid))$ satisfy the conditions (L1) \sim (L4) and we set $Q_{\mathfrak{h}} = \sum_{\alpha \in \Delta} \mathbb{Z}\alpha$ where $\Delta = \text{supp}_{\mathfrak{h}*}(\mathfrak{L}) \setminus \{0\}$. (a) Let $s \in \text{Hom}(Q_{\mathfrak{h}}, \Lambda)$. For a suitable bilinear form $(\mid)^{(s)}$, $(\mathfrak{L}^{(s)}, \mathfrak{h}, (\mid)^{(s)})$ also satisfies the conditions (L1) \sim (L4), and $\mathfrak{L} \sim_{\text{EALA}} \mathfrak{L}^{(s)}$. (b) Let $\rho : \langle \text{supp}_{\Lambda}(\mathfrak{L}) \rangle \rightarrow \Lambda$ be a monomorphism. For a suitable bilinear form $(\mid)_{(\rho)}$ on $\mathfrak{L}_{(\rho)}$, $(\mathfrak{L}_{(\rho)}, \mathfrak{h}, (\mid)_{(\rho)})$ also satisfies the conditions (L1) \sim (L4), and $\mathfrak{L} \sim_{\text{EALA}} \mathfrak{L}_{(\rho)}$.*

Proof. (a) Since $\mathfrak{L} = \mathfrak{L}^{(s)}$ as a Lie algebra, we can see (\mid) as a bilinear form on $\mathfrak{L}^{(s)}$. Let $(\mid)^{(s)}$ be this bilinear form. Then it is easily checked that $(\mathfrak{L}^{(s)}, \mathfrak{h}, (\mid)^{(s)})$ satisfies (L2) \sim (L4). To show that $\mathfrak{L}^{(s)}$ is graded-central-simple Λ -graded, suppose that $I \subseteq \mathfrak{L}^{(s)}$ is a Λ -graded ideal. Then I is $Q_{\mathfrak{h}} \times \Lambda$ -graded by (L4). By considering I as a ideal of \mathfrak{L} , we can see that $I = \{0\}$ or $\mathfrak{L}^{(s)}$. Also $C(\mathfrak{L}^{(s)})^0 = k \cdot \text{id}$ is clear,

and $\mathfrak{L}^{(s)}$ satisfies (L1). The second statement can be proved by the exactly same way as [AF, Corollary 6.3].

(b) Since $\mathfrak{L} = \mathfrak{L}_{(\rho)}$ as a Lie algebra, we can see an identity on \mathfrak{L} as an isomorphism from \mathfrak{L} onto $\mathfrak{L}_{(\rho)}$. We denote this isomorphism by $\psi : \mathfrak{L} \rightarrow \mathfrak{L}_{(\rho)}$, and define a bilinear form $(\mid)_{(\rho)}$ on $\mathfrak{L}_{(\rho)}$ as $(\psi(x)\mid\psi(y))_{(\rho)} = (x\mid y)$ for $x, y \in \mathfrak{L}$. The central grading group of $\mathfrak{L}_{(\rho)}$ is $\rho(\Gamma)$. Thus it is easily checked that $(\mathfrak{L}_{(\rho)}, \mathfrak{h}, (\mid)_{(\rho)})$ satisfies (L1) \sim (L4). To show that $\mathfrak{L} \sim_{\text{EALA}} \mathfrak{L}_{(\rho)}$, we first define a map

$$\mathcal{P}(\mathfrak{L}) \ni (\mathfrak{D}, \tau) \mapsto (\mathfrak{D}_{(\rho)}, \tau_{(\rho)}) \in \mathcal{P}(\mathfrak{L}_{(\rho)}).$$

As in the proof of Proposition 5.2.4, we write $C(\mathfrak{L}) = \bigoplus_{\mu \in \Gamma} k t^\mu$. Then we can write $C(\mathfrak{L}_{(\rho)}) = \bigoplus_{\mu \in \Gamma} k s^{\rho(\mu)}$ where

$$s^{\rho(\mu)} \cdot \psi(x) = \psi(t^\mu \cdot x)$$

for $\mu \in \Gamma, x \in \mathfrak{L}$. Let $\theta \in \text{Hom}(\Lambda, k)$. Since $\text{Im } \rho = \langle \text{supp}_\Lambda(\mathfrak{L}_{(\rho)}) \rangle$, we can define $\theta \circ \rho^{-1}$ as a homomorphism from $\langle \text{supp}_\Lambda(\mathfrak{L}_{(\rho)}) \rangle$ to k . Since the rank of $\langle \text{supp}_\Lambda(\mathfrak{L}_{(\rho)}) \rangle$ is n , there exists unique $\widetilde{\theta \circ \rho^{-1}} \in \text{Hom}(\Lambda, k)$ such that $\widetilde{\theta \circ \rho^{-1}}|_{\langle \text{supp}_\Lambda(\mathfrak{L}_{(\rho)}) \rangle} = \theta \circ \rho^{-1}$. Using this notation, we define a k -linear isomorphism $\omega : \text{SCDer}(\mathfrak{L}) \rightarrow \text{SCDer}(\mathfrak{L}_{(\rho)})$ as

$$\omega(t^\mu \partial_\theta) = s^{\rho(\mu)} \partial_{\widetilde{\theta \circ \rho^{-1}}}$$

for $\mu \in \Gamma, \theta \in \text{Hom}(\Lambda, k)$. Since

$$\begin{aligned} [s^{\rho(\mu_1)} \partial_{\widetilde{\theta_1 \circ \rho^{-1}}}, s^{\rho(\mu_2)} \partial_{\widetilde{\theta_2 \circ \rho^{-1}}}] &= \theta_1(\mu_2) s^{\rho(\mu_1 + \mu_2)} \partial_{\widetilde{\theta_2 \circ \rho^{-1}}} - \theta_2(\mu_1) s^{\rho(\mu_1 + \mu_2)} \partial_{\widetilde{\theta_1 \circ \rho^{-1}}} \\ &= \theta_1(\mu_2) \omega(t^{\mu_1 + \mu_2} \partial_{\theta_2}) - \theta_2(\mu_1) \omega(t^{\mu_1 + \mu_2} \partial_{\theta_1}), \end{aligned}$$

ω is a Lie algebra isomorphism. We have

$$\omega(d)(\psi(x)) = \psi(d(x)) \tag{37}$$

for $d \in \mathfrak{D}, x \in \mathfrak{L}$ since if $y \in \mathfrak{L}^\lambda$,

$$\begin{aligned} \omega(t^\mu \partial_\theta)(\psi(y)) &= s^{\rho(\mu)} \partial_{\widetilde{\theta \circ \rho^{-1}}}(\psi(y)) = \widetilde{\theta \circ \rho^{-1}}(\rho(\lambda)) s^{\rho(\mu)} \cdot \psi(y) \\ &= \theta(\lambda) \psi(t^\mu \cdot y) = \psi(t^\mu \partial_\theta(y)). \end{aligned}$$

We put $\mathfrak{D}_{(\rho)} = \omega(\mathfrak{D})$, and set $\mathfrak{C}_{(\rho)} = \bigoplus_{\mu \in \Gamma} (\mathfrak{D}_{(\rho)})^\mu$. We define $\hat{\omega} : \mathfrak{C} \rightarrow \mathfrak{C}_{(\rho)}$ by

$$\hat{\omega}(c)(\omega(d)) = c(d)$$

for $c \in \mathfrak{C}, d \in \mathfrak{D}$, and define $\tau_{(\rho)} : \mathfrak{D}_{(\rho)} \times \mathfrak{D}_{(\rho)} \rightarrow \mathfrak{C}_{(\rho)}$ as

$$\tau_{(\rho)}(\omega(d_1), \omega(d_2))(\omega(d_3)) = \tau(d_1, d_2)(d_3)$$

for $d_i \in \mathfrak{D}$. Then $(\mathfrak{D}_{(\rho)}, \tau_{(\rho)}) \in \mathcal{P}(\mathfrak{L}_{(\rho)})$ is clear. Next, we show that the map $x + c + d \mapsto \psi(x) + \hat{\omega}(c) + \omega(d)$ for $x \in \mathfrak{L}, c \in \mathfrak{C}, d \in \mathfrak{D}$ is a Lie algebra isomorphism. To prove this fact, it suffices to show that

$$\sigma_{\mathfrak{D}_{(\rho)}}(\psi(x_1), \psi(x_2)) = \hat{\omega}(\sigma_{\mathfrak{D}}(x_1, x_2)) \tag{38}$$

for $x_i \in \mathfrak{L}$ since we have using (37) that

$$\begin{aligned}
& [\psi(x_1) + \hat{\omega}(c_1) + \omega(d_1), \psi(x_2) + \hat{\omega}(c_2) + \omega(d_2)] \\
&= \left([\psi(x_1), \psi(x_2)] + \omega(d_1)(\psi(x_2)) - \omega(d_2)(\psi(x_1)) \right) \\
&+ \left(\sigma_{\mathfrak{D}(\rho)}(\psi(x_1), \psi(x_2)) + \omega(d_1) \cdot \hat{\omega}(c_2) - \omega(d_2) \cdot \hat{\omega}(c_1) + \tau_{(\rho)}(\omega(d_1), \omega(d_2)) \right) \\
&+ [\omega(d_1), \omega(d_2)] \\
&= \left(\psi([x_1, x_2]) + \psi(d_1(x_2)) - \psi(d_2(x_1)) \right) \\
&+ \left(\sigma_{\mathfrak{D}(\rho)}(\psi(x_1), \psi(x_2)) + \hat{\omega}(d_1 \cdot c_2) - \hat{\omega}(d_2 \cdot c_1) + \hat{\omega}(\tau(d_1, d_2)) \right) + \omega([d_1, d_2]).
\end{aligned}$$

(38) follows since

$$\begin{aligned}
\sigma_{\mathfrak{D}(\rho)}(\psi(x_1), \psi(x_2))(\omega(d)) &= (\omega(d)(\psi(x_1)) | \psi(x_2))_{(\rho)} = (\psi(d(x_1)) | \psi(x_2))_{(\rho)} \\
&= (d(x_1) | x_2) = \sigma_{\mathfrak{D}}(x_1, x_2)(d).
\end{aligned}$$

It is easy to see that the isomorphism preserve a bilinear form and sends H to $H_{(\rho)} = \mathfrak{h} \oplus \mathfrak{C}_{(\rho)} \oplus \mathfrak{D}_{(\rho)}$. Finally, to show that the map $(\mathfrak{D}, \tau) \mapsto (\mathfrak{D}_{(\rho)}, \tau_{(\rho)})$ is bijective, we construct the converse of this map. By (27), ρ induces a group isomorphism $\bar{\rho} : \langle \text{supp}_{\Lambda}(\mathfrak{L}) \rangle \rightarrow \langle \text{supp}_{\Lambda}(\mathfrak{L}_{(\rho)}) \rangle$. For the canonical monomorphism $\iota : \langle \text{supp}_{\Lambda}(\mathfrak{L}_{(\rho)}) \rangle \rightarrow \Lambda$, it is easily checked that

$$(\mathfrak{L}_{(\rho)})_{(\iota \circ \bar{\rho}^{-1})} = \mathfrak{L}.$$

Then we can see that

$$\mathcal{P}(\mathfrak{L}_{(\rho)}) \ni (\mathfrak{D}', \tau') \mapsto (\mathfrak{D}'_{(\iota \circ \bar{\rho}^{-1})}, \tau'_{(\iota \circ \bar{\rho}^{-1})}) \in \mathcal{P}(\mathfrak{L})$$

is the converse of the map $(\mathfrak{D}, \tau) \mapsto (\mathfrak{D}_{(\rho)}, \tau_{(\rho)})$. \square

Let $\mathfrak{L} = L_{\mathbf{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ be a multiloop Lie algebra of nullity n such that $\mathfrak{g}^{\sigma} \neq \{0\}$. Let $(|)$ be the Killing form of \mathfrak{g} , and we define a non-degenerate, invariant, symmetric, \mathbb{Z}^n -graded bilinear form (we also write $(|)$) as

$$(x \otimes t^{\lambda} | y \otimes t^{\mu}) = \begin{cases} (x | y) & \text{if } \lambda + \mu = 0 \\ 0 & \text{otherwise} \end{cases} \quad (39)$$

where $\lambda, \mu \in \mathbb{Z}^n, x \in \mathfrak{g}^{\bar{\lambda}}, y \in \mathfrak{g}^{\bar{\mu}}$. Then, $(\mathfrak{L}, \mathfrak{h}, (|))$ satisfies (L1) \sim (L4) by Lemma 2.1.5 and Corollary 3.3.6. The following proposition shows that a bilinear form on \mathfrak{L} satisfying (L3) is only that defined in (39) up to a scalar multiplication.

Proposition 5.2.6. *Suppose that a bilinear form $(|)'$ on \mathfrak{L} is non-degenerate, invariant, symmetric, and \mathbb{Z}^n -graded. Then we have $(|)' = c(|)$ for $0 \neq c \in k$, where $(|)$ is the bilinear form defined in (39).*

Proof. We write $xt^{\lambda} = x \otimes t^{\lambda} \in \mathfrak{L}$. For each $\alpha \in \Delta$, we take a $\mathfrak{sl}_2(k)$ -triple $\{x_{\alpha}^{\bar{\lambda}_{\alpha}}, x_{-\alpha}^{-\bar{\lambda}_{\alpha}}, h_{\alpha}\}$ for some $\lambda_{\alpha} \in \mathbb{Z}^n$. We choose $\gamma \in \Delta$ arbitrarily, and suppose that

$(h_\gamma|h_\gamma)' = c(h_\gamma|h_\gamma)$. If $\beta \in \Delta$ satisfies $(h_\beta|h_\gamma) \neq 0$,

$$\begin{aligned} (h_\beta|h_\beta)' &= (h_\beta|[x_\beta^{\bar{\lambda}_\beta} t^{\lambda_\beta}, x_{-\beta}^{-\bar{\lambda}_\beta} t^{-\lambda_\beta}])' = 2(x_\beta^{\bar{\lambda}_\beta} t^{\lambda_\beta} | x_{-\beta}^{-\bar{\lambda}_\beta} t^{-\lambda_\beta})' \\ &= \frac{2}{\langle \beta, h_\gamma \rangle} ([h_\gamma, x_\beta^{\bar{\lambda}_\beta} t^{\lambda_\beta}] | x_{-\beta}^{-\bar{\lambda}_\beta} t^{-\lambda_\beta})' = \frac{(h_\beta|h_\beta)}{(h_\beta|h_\gamma)} (h_\gamma|h_\beta)' \\ &= \frac{(h_\beta|h_\beta)}{(h_\beta|h_\gamma)} ([x_\gamma^{\bar{\lambda}_\gamma} t^{\lambda_\gamma}, x_{-\gamma}^{-\bar{\lambda}_\gamma} t^{-\lambda_\gamma}] | h_\beta)' = \frac{2(h_\beta|h_\beta)}{(h_\gamma|h_\gamma)} (x_\gamma^{\bar{\lambda}_\gamma} t^{\lambda_\gamma} | x_{-\gamma}^{-\bar{\lambda}_\gamma} t^{-\lambda_\gamma})' \\ &= \frac{(h_\beta|h_\beta)}{(h_\gamma|h_\gamma)} (h_\gamma|h_\gamma)' = c(h_\beta|h_\beta). \end{aligned}$$

By repeating calculations as above, we have $(h_\alpha|h_\alpha)' = c(h_\alpha|h_\alpha)$ for $\alpha \in \Delta$ since Δ is irreducible. Then for arbitrary $\alpha \in \Delta, \lambda \in \mathbb{Z}^n$ and $x \in \mathfrak{g}_\alpha^{\bar{\lambda}}, y \in \mathfrak{g}_{-\alpha}^{-\bar{\lambda}}$,

$$\begin{aligned} (xt^\lambda | yt^{-\lambda})' &= \frac{1}{2} ([h_\alpha, xt^\lambda] | yt^{-\lambda})' = \frac{1}{2} (h_\alpha | [xt^\lambda, yt^{-\lambda}])' \\ &= \frac{(x|y)}{2} (h_\alpha | \frac{(\alpha|\alpha)}{2} h_\alpha)' \quad (\text{by (17) and (18)}) \\ &= c(x|y) = c(xt^\lambda | yt^{-\lambda}). \end{aligned}$$

From this, we have $(\cdot | \cdot)' = c(\cdot | \cdot)$ on $\bigoplus_{\alpha \in \Delta} \mathfrak{L}_\alpha$. Then we have $(\cdot | \cdot)' = c(\cdot | \cdot)$ on \mathfrak{L} since $\mathfrak{L}_0 \subseteq \bigoplus_{\alpha \in \Delta} [\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]$ and both $(\cdot | \cdot)$ and $(\cdot | \cdot)'$ are invariant. \square

Remark 5.2.7. Suppose that \mathfrak{L} is a multiloop Lie algebra and $(\mathfrak{D}, \tau) \in \mathcal{P}(\mathfrak{L})$. By Proposition 5.2.6, it is easily checked that $E(\mathfrak{L}, \mathfrak{D}, \tau)$ does not depend on the bilinear form used in the construction up to isomorphism as EALAs.

Now, we can easily show the following theorem:

Theorem 5.2.8. *Let $\mathfrak{L} = L_{\mathfrak{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ and $\mathfrak{L}' = L_{\mathfrak{m}'}(\mathfrak{g}', \sigma', \mathfrak{h}')$ be multiloop Lie algebras of nullity n , and suppose that $\mathfrak{g}^\sigma \neq \{0\}$ and $\mathfrak{g}'^{\sigma'} \neq \{0\}$. If $\mathfrak{L} \cong_{\text{supp}} \mathfrak{L}'$, then there exists a bijection $\mathcal{P}(\mathfrak{L}) \rightarrow \mathcal{P}(\mathfrak{L}')$ such that $E(\mathfrak{L}, \mathfrak{D}, \tau)$ is isomorphic as EALAs to $E(\mathfrak{L}', \mathfrak{D}', \tau')$ where $(\mathfrak{D}', \tau') \in \mathcal{P}(\mathfrak{L}')$ is the image of $(\mathfrak{D}, \tau) \in \mathcal{P}(\mathfrak{L})$ under the bijection.*

Proof. By Theorem 4.2.5 and Lemma 5.2.5, there exists a $Q_{\mathfrak{h}} \times \mathbb{Z}^n$ -graded Lie algebra \mathfrak{L}_p such that $\mathfrak{L}_p \cong_{\mathbb{Z}^n\text{-ig}} \mathfrak{L}'$ and $\mathfrak{L} \sim_{\text{EALA}} \mathfrak{L}_p$. Using Lemma 4.1.3, it is easily checked that $\mathfrak{L}_p \sim_{\text{EALA}} \mathfrak{L}'$. Thus we have $\mathfrak{L} \sim_{\text{EALA}} \mathfrak{L}'$. \square

We prove the following lemma using [AF, Theorem 6.1]:

Lemma 5.2.9. *Let \mathcal{L} and \mathcal{L}' be multiloop Lie \mathbb{Z}^n -tori. If $E(\mathcal{L}, \mathfrak{D}, \tau)$ is isomorphic as EALAs to $E(\mathcal{L}', \mathfrak{D}', \tau')$ for some $(\mathfrak{D}, \tau) \in \mathcal{P}(\mathcal{L})$ and $(\mathfrak{D}', \tau') \in \mathcal{P}(\mathcal{L}')$, then $\mathcal{L} \cong_{\text{supp}} \mathcal{L}'$.*

Proof. Let Q (resp. Q') be a root lattice of \mathcal{L} (resp. \mathcal{L}'). By [AF, Theorem 6.1], there exists $s \in \text{Hom}(Q, \mathbb{Z}^n)$, a Lie algebra isomorphism $\varphi : \mathcal{L}^{(s)} \rightarrow \mathcal{L}'$ and two group isomorphisms $\varphi_Q : Q \rightarrow Q', \varphi_{\mathbb{Z}^n} : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ such that

$$\varphi((\mathcal{L}^{(s)})_\alpha^\lambda) = \mathcal{L}'^{\varphi_{\mathbb{Z}^n}(\lambda)}_{\varphi_Q(\alpha)}$$

for $\alpha \in Q, \lambda \in \mathbb{Z}^n$ (in [AF], this equivalence relation is called to be isotopic). Then $\mathcal{L} \cong_{\text{supp}} \mathcal{L}'$ follows. (See the proof of Theorem 4.2.5 (c) \Rightarrow (a).) \square

Using Theorem 5.1.4, we can extend this lemma to multiloop Lie algebras of which 0-homogeneous spaces are non-zero.

Theorem 5.2.10. *Let $\mathfrak{L} = L_{\mathfrak{m}}(\mathfrak{g}, \sigma, \mathfrak{h})$ and $\mathfrak{L}' = L_{\mathfrak{m}'}(\mathfrak{g}', \sigma', \mathfrak{h}')$ be multiloop Lie algebras of nullity n , and suppose that $\mathfrak{g}^\sigma \neq \{0\}$ and $\mathfrak{g}'^{\sigma'} \neq \{0\}$. If $E(\mathfrak{L}, \mathfrak{D}, \tau)$ is isomorphic as EALAs to $E(\mathfrak{L}', \mathfrak{D}', \tau')$ for some $(\mathfrak{D}, \tau) \in \mathcal{P}(\mathfrak{L})$ and $(\mathfrak{D}', \tau') \in \mathcal{P}(\mathfrak{L}')$, then $\mathfrak{L} \cong_{\text{supp}} \mathfrak{L}'$.*

Proof. By Theorem 5.1.4, there exists a multiloop Lie \mathbb{Z}^n -torus \mathcal{L} (resp. \mathcal{L}') such that $\mathfrak{L} \cong_{\text{supp}} \mathcal{L}$ (resp. $\mathfrak{L}' \cong_{\text{supp}} \mathcal{L}'$). Then by Theorem 5.2.8, there exists $(\mathfrak{D}, \bar{\tau}) \in \mathcal{P}(\mathcal{L})$ (resp. $(\mathfrak{D}', \bar{\tau}') \in \mathcal{P}(\mathcal{L}')$) such that $E(\mathfrak{L}, \mathfrak{D}, \tau)$ and $E(\mathcal{L}, \mathfrak{D}, \bar{\tau})$ (resp. $E(\mathfrak{L}', \mathfrak{D}', \tau')$ and $E(\mathcal{L}', \mathfrak{D}', \bar{\tau}')$) are isomorphic as EALAs. Therefore, $E(\mathcal{L}, \mathfrak{D}, \bar{\tau})$ and $E(\mathcal{L}', \mathfrak{D}', \bar{\tau}')$ are isomorphic as EALAs, and then $\mathcal{L} \cong_{\text{supp}} \mathcal{L}'$ by Lemma 5.2.9. Thus, we have $\mathfrak{L} \cong_{\text{supp}} \mathfrak{L}'$. \square

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REFERENCES

- [ABFP1] B. Allison, S. Berman, J. Faulkner, A. Pianzola, *Realization of graded- simple algebras as loop algebras*, arXiv:math/0511723v2 [math.RA].
- [ABFP2] B. Allison, S. Berman, J. Faulkner and A. Pianzola, *Multiloop realization of extended affine Lie algebras and Lie tori*, arXiv:0709.0975v2 [math.RA].
- [AF] B. Allison and J. Faulkner, *Isotopy for extended affine Lie algebras and Lie tori*, arXiv:0709.1181v3 [math.RA].
- [B1] N. Bourbaki, *Lie groups and Lie algebras. Chapters 1-3*, Translated from the French, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1989.
- [B2] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4-6*, Translated from the French, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002.
- [BN] G. Benkart and E. Neher, *The centroid of extended affine and root graded Lie algebras*, J. Pure Appl. Algebra, **205** (2006), pp. 117-145.
- [H] J. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, 1972.
- [K] V.G. Kac, *Infinite dimensional Lie algebras*, Third edition, Cambridge University Press, Cambridge, 1990.
- [N] E. Neher, *Extended affine Lie algebras*, C.R. Math. Acad. Sci. Soc. R. Can., **26** (2004), pp. 90-96.

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